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Uniformly resolvable designs with index one and block sizes three and four — with three or five parallel classes of block size four

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ABSTRACT

Each parallel class of a uniformly resolvable design (URD) contains blocks of only one block size. A URD with v points and with block sizes three and four means that at least one parallel class has block size three and at least one has block size four. Danziger [P. Danziger, Uniform restricted resolvable designs with $r = 3$, ARS Combin. 46 (1997) 161–176] proved that for all $v \equiv 12 \pmod{24}$ there exist URDs with index one, some parallel classes of block size three, and exactly three parallel classes with block size four, except when $v = 12$ and except possibly when $v = 84\,156$. We extend Danziger's work by showing that there exists a URD with index one, some parallel classes with block size three, and exactly three parallel classes with block size four if, and only if, $v \equiv 0 \pmod{12}$, $v \neq 12$. We also prove that there exists a URD with index one, some parallel classes of block size three, and exactly five parallel classes with block size four if, and only if, $v \equiv 0 \pmod{12}$, $v \neq 12$. New labeled URDs, which give new URDs as ingredient designs for recursive constructions, are the key in the proofs. Some ingredient URDs are also constructed with difference families.

1. Introduction

Let v and λ be positive integers, and let K and M be two sets of positive integers. A *group divisible design*, denoted $\mathrm{GDD}_\lambda(K, M; v)$, is a triple $(X, \boldsymbol{G}, \boldsymbol{B})$, where X is a set with v elements (called points), $\boldsymbol{G}$ is a set of subsets (called groups) of X, $\boldsymbol{G}$ partitions X, and $\boldsymbol{B}$ is a collection of subsets (called blocks) of X such that:

1. $|B| \in K$ for each $B \in \boldsymbol{B}$;
2. $|G| \in M$ for each $G \in \boldsymbol{G}$;
3. $|B \cap G| \leq 1$ for each $B \in \boldsymbol{B}$ and each $G \in \boldsymbol{G}$; and
4. each pair of elements of X from distinct groups is contained in exactly λ blocks.

The notation is similar to that used in [1]. Unless otherwise stated, the element set X of a design with v points is labeled $1, 2, \ldots, v$. If $\lambda = 1$, the index λ is omitted. If $K = \{k\}$ (respectively $M = \{m\}$) then the $\mathrm{GDD}_\lambda(k, M; v)$ is simply denoted $\mathrm{GDD}_\lambda(k, M; v)$ (respectively $\mathrm{GDD}_\lambda(K, m; v)$). A $\mathrm{GDD}_\lambda(K, 1; v)$ is called a *pairwise balanced design* and denoted $\mathrm{PBD}_\lambda(K; v)$.

In a $\mathrm{GDD}_\lambda(K, M; v)(X, \boldsymbol{G}, \boldsymbol{B})$, a *parallel class* is a set of blocks of $\boldsymbol{B}$ which partitions X. If $\boldsymbol{B}$ can be partitioned into parallel classes, then the $\mathrm{GDD}_\lambda(K, M; v)$ is called *resolvable* and denoted $\mathrm{RGDD}_\lambda(K, M; v)$. Analogously, a resolvable $\mathrm{PBD}_\lambda(K; v)$ is denoted $\mathrm{RPBD}_\lambda(K; v)$. A parallel class is called *uniform* if it contains blocks of only one size. If all parallel classes of an $\mathrm{RPBD}_\lambda(K; v)$ ($\mathrm{RGDD}_\lambda(K, M; v)$) are uniform, the design is called *uniformly resolvable*. Here, a uniformly resolvable design $\mathrm{RPBD}_\lambda(K; v)$ ($\mathrm{RGDD}_\lambda(K, M; v)$) is denoted $\mathrm{URD}_\lambda(K; v)$ ($\mathrm{UGDD}_\lambda(K, M; v)$). If $\lambda = 1$, the index λ is omitted. In a $\mathrm{URD}_\lambda(K; v)$ ($\mathrm{UGDD}_\lambda(K, M; v)$) the number of parallel classes with blocks of size k, $k \in K$, is denoted r_k. A *resolvable transversal design*, denoted $\mathrm{RTD}_\lambda(k, g)$, is equivalent to an $\mathrm{RGDD}_\lambda(k, g; k \cdot g)$. That is, each block in an $\mathrm{RTD}_\lambda(k, g)$ contains a point from each

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group. A K-frame$_\lambda$ is a GDD $(X, \boldsymbol{G}, \boldsymbol{B})$ with index λ in which the collection of blocks $\boldsymbol{B}$ can be partitioned into holey parallel classes each of which partitions $X \setminus G$ for some $G \in \boldsymbol{G}$. We use the usual exponential notation for the types of GDDs and frames. Thus a GDD or a frame of type $1^i 2^j \ldots$ is one in which there are i groups of size 1, j groups of size 2, and so on. A K-frame is called *uniform* if each partial parallel class is of only one block size. It is called *completely uniform* if for each hole G the resolution classes which partition $X \setminus G$ are all of one block size. Due to the fact that the block size is often 3 or 4 in this paper, set $\hat{K} = \{3, 4\}$. Uniformly resolvable designs with block sizes three and four mean here URD$(\hat{K}; v)$ with $r_3, r_4 > 0$. Firstly, the necessary conditions for these designs are given.

Theorem 1.1 *([3], Theorems 1.1 and 3.5). The necessary conditions for the existence of a URD$(\hat{K}; v)$ with $r_3, r_4 > 0$ are:*

- $v \equiv 0 \pmod{12}$*;*
- r_4 *is odd;*
- *if* $r_k > 1$*, then* $v \geq k^2$*; and*
- $r_4 = \frac{v-1-2\cdot r_3}{3}$ $\left(r_3 = \frac{v-1-3\cdot r_4}{2}\right)$.

The fourth condition means that if r_3 is given, then r_4 is determined, and vice versa. We now give some well-known results.

Theorem 1.2 *([6]). There exists an RGDD$(3, 4; v)$ and also a URD$(\hat{K}; v)$ with $r_4 = 1$ if, and only if, $v \equiv 0 \pmod{12}$.*

Take the groups of the RGDD as an additional parallel class to get the URD.

Theorem 1.3 *([4,7,9,11]). There exists an RGDD$(4, 3; v)$ and also a URD$(\hat{K}; v)$ with $r_3 = 1$ if, and only if, $v \equiv 0 \pmod{12}$, $v \geq 24$.*

Theorem 1.4. *There exists a URD$(\hat{K}; v)$ with $r_4 = 3$ for all $v \equiv 12 \pmod{24}$, $v \geq 36$.*

Proof. It is shown in [2] that the required design exists for all $v \equiv 12 \pmod{24}$, $v \neq 12$, and with the possible exceptions of $v = 84, 156$. We completely settle this problem by providing a URD$(\hat{K}; 84)$ and a URD$(\hat{K}; 156)$ both with $r_4 = 3$ in the Appendix. □

Simple counting arguments show that the following conditions are necessary for the existence of an RGDD$(k, m; v)$: $v \equiv 0 \pmod{m}$, $v \geq k \cdot m$, $v \equiv 0 \pmod{k}$, $v - m \equiv 0 \pmod{(k-1)}$. The necessary conditions for the existence of an RGDD$(3, m; v)$ are also sufficient with three exceptions:

Theorem 1.5 *([6,11]). There exists an RGDD$(3, m; v)$ if, and only if, $v \equiv 0 \pmod{m}$, $v \geq 3 \cdot m$, $v \equiv 0 \pmod{3}$ and $v - m \equiv 0 \pmod{2}$, except when $(v, m) = (6, 2), (12, 2), (18, 3)$.*

In the next section, labeled resolvable designs are introduced. Ingredient designs for recursive constructions, which are described in section three, are created by some new labeled uniformly resolvable designs. The fourth section contains results for URD$(\hat{K}; v)$ with $r_4 = 3$ and $v \equiv 0 \pmod{24}$, and in the last section there are results for URD$(\hat{K}; v)$ with $r_4 = 5$.

2. Labeled resolvable designs

We use the concept of labeled resolvable designs in order to get direct constructions for resolvable designs. This concept was introduced by Shen; see [8,10,11].

Let $(X, \boldsymbol{G}, \boldsymbol{B})$ be a (U)GDD$_\lambda(K, G; v)$ where $X = \{a_1, a_2, \ldots, a_v\}$ is totally ordered with ordering $a_1 < a_2 < \cdots < a_v$. For each block $B = \{x_1, x_2, \ldots, x_k\}$, $k \in K$, it is supposed that $x_1 < x_2 < \cdots < x_k$. Let Z_λ be the group of residues modulo λ.

Let $\varphi : \boldsymbol{B} \to Z_\lambda^{\binom{k}{2}}$ be a mapping where for each $B = \{x_1, x_2, \ldots, x_k\} \in \boldsymbol{B}$, $k \in K$,

$$\varphi(B) = (\varphi(x_1, x_2), \ldots, \varphi(x_1, x_k), \varphi(x_2, x_3), \ldots, \varphi(x_2, x_k), \varphi(x_3, x_4), \ldots, \varphi(x_{k-1}, x_k)), \quad \varphi(x_i, x_j) \in Z_\lambda$$

for $1 \leq i < j \leq k$.

A (U)GDD$_\lambda(K, G; v)$ is said to be a *labeled (uniform resolvable) group divisible design*, denoted L(U)GDD$_\lambda(K, G; v)$, if there exists a mapping φ such that:

1. For each pair $\{x, y\} \subset X$ with $x < y$, contained in the blocks $B_1, B_2, \ldots, B_\lambda$, then $\varphi_i(x, y) \equiv \varphi_j(x, y) \pmod{\lambda}$ if, and only if, $i = j$ where the subscripts i and j denote the blocks to which the pair belongs, for $1 \leq i, j \leq \lambda$;
2. For each block $B = \{x_1, x_2, \ldots, x_k\}$, $k \in K$, $\varphi(x_r, x_s) + \varphi(x_s, x_t) \equiv \varphi(x_r, x_t) \pmod{\lambda}$, for $1 \leq r < s < t \leq k$.

Its blocks will be denoted in the following form:

$$(x_1\, x_2 \ldots x_k; \varphi(x_1, x_2) \ldots \varphi(x_1, x_k)\varphi(x_2, x_3) \ldots \varphi(x_2, x_k)\varphi(x_3, x_4) \ldots \varphi(x_{k-1}, x_k)), \quad k \in K.$$

The above definition is a little bit more general than the definition by Shen [11] with $K = \{k\}$ or Shen and Wang [10] for transversal designs. As special case of type 1^v, a labeled URD$_\lambda(K; v)$ is denoted LURD$_\lambda(K; v)$.

Example 2.1. The following is an example of an $LURD_3(\hat{K}; 12)$ with $r_4 = 5$, where each row forms a parallel class:

(3 6 9; 2 1 2), (1 8 11; 0 1 1), (4 7 10; 2 0 1), (2 5 12; 0 1 1),
(2 7 11; 0 1 1), (1 6 12; 0 2 2), (4 5 9; 2 2 0), (3 8 10; 0 1 1),
(3 4 6; 0 1 1), (2 5 12; 1 0 2), (1 7 9; 2 1 2), (8 10 11; 2 0 1),
(4 6 8; 2 2 0), (1 9 10; 2 2 0), (3 5 11; 2 0 1), (2 7 12; 1 2 1),
(5 9 10; 1 0 2), (2 6 11; 0 2 2), (1 3 7; 2 0 1), (4 8 12; 0 0 0),
(2 4 11; 2 0 1), (3 9 12; 0 1 1), (1 5 8; 1 2 1), (6 7 10; 1 0 2),
(7 8 9; 2 0 1), (1 6 11; 1 2 1), (2 4 5; 1 2 1), (3 10 12; 0 0 0),
(1 2 9; 2 0 1), (8 11 12; 2 2 0), (4 6 10; 0 1 1), (3 5 7; 0 2 2),
(2 9 10; 0 1 1), (3 4 11; 2 2 0), (1 6 12; 2 0 1), (5 7 8; 1 2 1),
(1 3 4 5; 1 2 2 1 1 0), (2 6 8 9; 1 2 2 1 1 0), (7 10 11 12; 0 0 2 0 2 2),
(5 6 9 11; 2 2 2 0 0 0), (1 4 7 12; 0 1 1 1 1 0), (2 3 8 10; 1 0 0 2 2 0),
(3 9 11 12; 2 1 2 2 0 1), (5 6 7 8; 1 0 0 2 2 0), (1 2 4 10; 1 1 0 0 2 2),
(1 5 10 11; 0 1 0 1 0 2), (4 8 9 12; 1 0 2 2 1 2), (2 3 6 7; 2 2 2 0 0 0),
(5 6 10 12; 0 2 0 2 0 1), (4 7 9 11; 0 1 2 1 2 1), (1 2 3 8; 0 0 1 0 1 1).

Further examples are given in the Appendix. The main application of the labeled designs is to blow up the point set of a given design using the following theorem, which extends the work of [8] such that it is applicable to labeled (uniform resolvable) pairwise balanced designs.

Theorem 2.2. *If there exists a $L(U)GDD_\lambda(K, G; v)$ (with r_k^L classes of size k, for each $k \in K$), then there exists a $(U)GDD(K, \lambda \cdot G; \lambda \cdot v)$, where $\lambda \cdot G = \{\lambda \cdot g_i | g_i \in G\}$ (with $r_k = r_k^L$ classes of size k, for each $k \in K$).*

Proof. Let $(X, \boldsymbol{G}, \boldsymbol{B})$ be an $LRGDD_\lambda(K, G; v)$ where $X = \{a_1, a_2, \dots, a_v\}$. Expanding each point $a_i \in X$ λ times gives the points $\{a_{i,0}, \dots, a_{i,\lambda-1}\}$, $i = 1, \dots, v$, in the new design. Any group with g_i points becomes a new group with $\lambda \cdot g_i$ points. Each labeled block

$$(x_1\ x_2 \dots x_k;\ \varphi(x_1, x_2) \dots \varphi(x_1, x_k)\varphi(x_2, x_3) \dots \varphi(x_2, x_k)\varphi(x_3, x_4) \dots \varphi(x_{k-1}, x_k)), \quad k \in K,$$

gives λ new blocks $\{x_{1,j}, x_{2,j+\varphi(x_1,x_2)}, \dots, x_{k,j+\varphi(x_1,x_k)}\}$, $k \in K, j = 0, \dots, (\lambda - 1)$, with indices calculated mod (λ) and where all blocks taken together consist of different points. Therefore, each parallel class of the labeled design with blocks of size k gives a parallel class of the expanded design with blocks of the same size k. For each pair $\{x, y\} \subset X$ with $x < y$ from different groups, let $B_1, B_2, \dots, B_\lambda$ be the λ blocks containing $\{x, y\}$ and let $\varphi_i(x, y)$ be the values of $\varphi(x, y)$ corresponding to B_i, $1 \le i \le \lambda$. Due to the first condition all pairs $\{x_j, y_{j+\varphi_i(x,y)}\}$, $i = 1, \dots, \lambda, j = 0, \dots, (\lambda - 1)$, with indices calculated modulo λ, are different. □

A special case for URDs is shown in the following.

Corollary 2.3. *If there exists an $LURD_\lambda(K; v)$ with r_k^L classes of size k, for each $k \in K$, then there exists a $URD(K \cup \{\lambda\}; \lambda \cdot v)$ with $r_k = r_k^L$ when $k \neq \lambda$, and $r_\lambda = r_\lambda^L + 1$, where we take $r_\lambda^L = 0$ if $\lambda \notin K$.*

Lemma 2.4. *An $LURD_\lambda(\hat{K}; v)$ exists for $(\lambda, v, r_4) \in \{(4, 12, 2), (3, 24, 3), (4, 24, 2)\}$.*

Proof. The designs are all given in the Appendix. □

Lemma 2.5. *A $URD(\hat{K}; v)$ with $r_4 = 3$ exists for $v \in \{24, 48, 72, 84, 96, 156\}$.*

Proof. The cases $v = 24$, 84 and 156 are given in the Appendix. For the remaining cases use Corollary 2.3, using a $LURD_\lambda(\hat{K}; v)$ from Lemma 2.4. Specifically $(\lambda, v, r_4) \in \{(4, 12, 2), (3, 24, 3), (4, 24, 2)\}$ gives the required designs for $v = 48, 72$ and 96, respectively. □

Lemma 2.6. *A $URD(\hat{K}; v)$ with $r_4 = 5$ exists for $v \in \{24, 36, 84, 132, 156, 204\}$.*

Proof. The cases $v = 24, 84, 132, 156$ and 204 are given in the Appendix. For $v = 36$, use Corollary 2.3 with an $LURD_3(\hat{K}; 12)$ with $r_4 = 5$ from Example 2.1. □

3. Constructions

We now describe some constructions which we use later. Firstly, some Wilson type constructions are shown, where each point of a *master* design is expanded and the resulting large blocks are filled with so-called *ingredient* designs.

Theorem 3.1. *There exists a $URD(\hat{K}; v)$ with $r_4 = 5$ for all $v \equiv 0 \pmod{48}$, $v \ge 48$.*

Proof. Let $v \equiv 0 \pmod{12}$. By Theorem 1.2 we can take as a master design a URD$(\hat{K}; v)$ with $r_4 = 1$. Expand all points of this master design four times. It is well known that an RGDD(3, 4; 12) and an RTD(4, 4) exist. For each block with $k = 3$ the expanded block is filled with an RGDD(3, 4; 12) where the expanded points become the groups. Each parallel class with $k = 3$ creates four new parallel classes with $k = 3$. For each block with $k = 4$ the expanded block is filled with an RTD(4, 4). The only parallel class with $k = 4$ creates four new parallel classes with $k = 4$. The groups of the new design give the fifth parallel class with $k = 4$. □

Theorem 3.2. *There exists a URD$(\hat{K}; v)$ with $r_4 = 5$ for all $v \equiv 0 \pmod{60}$, $v \geq 60$.*

Proof. Let $v \equiv 0 \pmod{12}$. By Theorem 1.2 we can take as a master design a URD$(\hat{K}; v)$ with $r_4 = 1$. Expand all points of this master design five times. It is well known that an RPBD(3; 15), an RTD(3, 5) and an RTD(4, 5) exist. For only one parallel class of size $k = 3$ fill each expanded block with an RPBD(3; 15), which also fills all pairs in the expanded groups. For all other parallel classes with $k = 3$ fill each expanded block with an RTD(3, 5). For the only parallel class with $k = 4$ fill each expanded block with an RTD(4, 5), which gives five parallel classes with $k = 4$. □

Theorem 3.3. *If there exists a URD$(\hat{K}; v)$ with $r_3 > 0$, $r_4 = m > 0$, then there exists a URD$(\hat{K}; n \cdot v)$ with $r_4 = m$, $n \geq 3$.*

Proof. Suppose that there exists a URD$(\hat{K}; v)$ with $r_3 > 0$ and $r_4 = m > 0$. Then $v \equiv 0 \pmod{12}$ and there exists a 3-RGDD of type v^n by Theorem 1.5. Filling the groups with the given URD, gives the desired URD. □

Theorem 3.3 is a special case of Lemma 2.5 in [3].

Theorem 3.4. *For $v \equiv 0 \pmod{24}$, $v \geq 72$, there exist a URD$(\hat{K}; v)$ with $r_4 = 3$ and a URD$(\hat{K}; v)$ with $r_4 = 5$.*

Proof. There exist a URD$(\hat{K}; 24)$ with $r_4 = 3$ and a URD$(\hat{K}; 24)$ with $r_4 = 5$ by Lemmas 2.5 and 2.6, respectively. Therefore, the assertion follows by Theorem 3.3. □

Theorem 3.5. *There exists a URD$(\hat{K}; v)$ with $r_4 = 5$, for all $v \equiv 0 \pmod{36}$, $v \geq 108$.*

Proof. A URD$(\hat{K}; 36)$ with $r_4 = 5$ exists by Lemma 2.6. Hence, the assertion follows by Theorem 3.3. □

Theorem 3.6. *If there exist an RPBD$(t; v)$ and an LURD$_\lambda(K; t \cdot \gamma)$ and some LUGDD$_\lambda(K, \gamma; t \cdot \gamma)$, then there exists an LURD$_\lambda(K; v \cdot \gamma)$ with r_k^L, $k \in K$ and therefore a URD$(K; v \cdot \gamma \cdot \lambda)$ with $r_k = r_k^L$ when $k \neq \lambda$, and $r_\lambda = r_\lambda^L + 1$, where we take $r_\lambda^L = 0$ if $\lambda \notin K$.*

Proof. We can take as a master design a RPBD$(t; v)$. Expand all points of this master design γ times. For only one parallel class each expanded block is filled with an LURD$_\lambda(K; t \cdot \gamma)$, this filled the pairs within the expanded points. For all other parallel classes each expanded block is filled with an LUGDD$_\lambda(K, \gamma; t \cdot \gamma)$. All blocks of any parallel class have to be filled with the same LUGDD$_\lambda(K, \gamma; t \cdot \gamma)$ (if more than one are given). Therefore, each parallel class expands in a way that several uniform parallel classes are created. Thus, the labeled expanded design is uniformly resolvable. The labeled property in this design is inherited from the labeled property of the ingredient designs. In a similar manner the uniform property is also inherited from the ingredients and the master design. The last assertion of Theorem 3.6 follows from Corollary 2.3. □

Theorem 3.7. *There exists an LURD$_4(\hat{K}; v)$ with $r_4 = 2$ for all $v \equiv 12 \pmod{24}$ and also a URD$(\hat{K}; v)$ with $r_4 = 3$, for all $v \equiv 48 \pmod{96}$.*

Proof. Let $v \equiv 3 \pmod{6}$. Take as a master design in Theorem 3.6 an RPBD$(3; v)$, which is well-known to exist; see [1]. Expand each point four times, that is choose $\gamma = 4$. As ingredient designs, take an LURD$_4(\hat{K}; 12)$ with $r_4 = 2$ and an LUGDD$_4(3, 4; 12)$, which are given in the Appendix. The assertion follows by Theorem 3.6. □

Theorem 3.7 gives new LURDs. The URDs can also be constructed with Theorem 3.1.

Theorem 3.8. *For $v \equiv 4 \pmod{12}$ there exists an LURD$_4(\hat{K}; 3 \cdot v)$ with $r_4 = 2, 4, \ldots, 2 \cdot (v-1)/3$ and also a URD$(\hat{K}; 12 \cdot v)$ with $r_4 = 3, 5, \ldots, 2 \cdot (v-1)/3 + 1$.*

Proof. Let $v \equiv 4 \pmod{12}$. Take as a master design in Theorem 3.6 an RPBD$(4; v)$, which is well-known to exist; see [1]. Expand each point three times, that is choose $\gamma = 3$. An LURD$_4(\hat{K}; 12)$ with $r_4 = 2$ is provided in the Appendix. An LUGDD$_4(3, 3; 12)$ and a LUGDD$_4(\hat{K}, 3; 12)$ with $r_4 = 2$ are also given in the Appendix. The assertion follows by Theorem 3.6. □

The following contains some further essentially needed constructions from [2].

Theorem 3.9 (*[2], Theorem 2.5*). *If there exists a uniform $\{3, 4\}$-frame of type $(g_1; 3^{\frac{g_1}{2}})^t (g_2; 3^{\frac{g_2 - 3 \cdot r}{2}} 4^r)^1$ and $w \equiv 3 \pmod{6}$ is such that $g_1 + w \equiv 3 \pmod{6}$, $2 \cdot w \leq g_1$, and there exists a URD$(\hat{K}; g_2 + w)$ with $r_4 = r$ $\left(r_3 = \frac{g_2 + w - 1 - 3 \cdot r}{2}\right)$, then there exists a URD$(\hat{K}; g_1 \cdot t + g_2 + w)$ with $r_4 = r$.*

Theorem 3.10 *([2], Lemmas 3.3 and 3.4). Let* $v_0 \equiv 0 \pmod{12}$, r_4 *odd.*

For $v_0 = 9 \cdot r_4 + 6 \cdot j + 9$ *with j and integer,* $j \geq 0$, *there exists a uniform* $\hat{K}$*-frame of type* $(24;\, 3^{12})^t (v_0 - 9;\, 3^{3(r_4+j)} 4^{r_4})^1$ *for all* $t \equiv 1 \pmod 3$ *with* $t \geq 1 + \frac{3\cdot(r_4+j)}{4}$.

For $v_0 = 9 \cdot r_4 + 6 \cdot j + 3$ *with j and integer,* $j \geq 0$, *there exists a uniform* $\hat{K}$*-frame of type* $(24;\, 3^{\cdots})^t (v_0 - 3;\, 3^{\cdots} 4^{r_4})^1$ *for all* $t \equiv 1 \pmod 3$ *with* $t \geq 1 + \frac{3\cdot(r_4+j)}{4}$.

Theorem 3.10 is a little bit more general than the lemmas in [2], but the proof is analogous. The second statement of Theorem 3.10 is only useful if $j = 0$. In all other cases the first variant is more effective, because the bound for t is lower.

For two values of v we need the following:

Construction 3.11 (*Weighting [4]*). *Let* $(X, \boldsymbol{G}, \boldsymbol{B})$ *be a GDD, and let* $w : X \to Z^+ \cup 0$ *be a weight function on X. Suppose that for each block* $B \in \boldsymbol{B}$, *there exists a k-frame of type* $\{w(x) : x \in B\}$. *Then there is a k-frame of type* $\left\{\sum_{x\in G_i} w(x) : G_i \in \boldsymbol{G}\right\}$.

4. Results for uniformly resolvable designs with block sizes three and four and exactly *three* parallel classes with block size four

Lemma 2.5 and Theorem 3.4 together with Theorem 1.4 result in:

Theorem 4.1. *There exists a URD*$(\{3, 4\};\, v)$ *with* $r_4 = 3$ *if, and only if,* $v \equiv 0 \pmod{12}$, $v \geq 24$.

5. Results for uniformly resolvable designs with block sizes three and four and exactly *five* parallel classes with block size four

Lemma 5.1. *There exists a URD*$(\hat{K};\, v)$ *with* $r_4 = 5$ *for all* $v \equiv 12 \pmod{72}$, $v \geq 84$.

Proof. A URD$(\hat{K};\, 60)$ with $r_4 = 5$ exists by Theorem 3.2. Since $60 = v_0 = 9 \cdot r_4 + 6 \cdot j + 9$, with $j = 1$, then by Theorem 3.10 there exists a $\hat{K}$-frame of type $(24;\, 3^{12})^t (v_0 - 9;\, 3^{18} 4^{r_4})^1$ for all $t \equiv 1 \pmod 3$ with $t \geq 1 + \frac{3\cdot(r_4+j)}{4} = 5.5$. Set $t = 4 + 3 \cdot i$ with $i = 1, 2, \ldots$. Therefore, by Theorem 3.9 there exist designs URD$(\hat{K};\, 24 \cdot (4 + 3 \cdot i) + 60)$ with $r_4 = 5$, $i = 1, 2, \ldots$. That is, all URD$(\hat{K};\, 156 + 72 \cdot i)$ with $r_4 = 5$, $i = 1, 2, \ldots$ exist.

Due to the third condition of Theorem 1.1 a URD$(\hat{K};\, 12)$ with $r_4 = 5$ cannot exist.

A URD$(\hat{K};\, 84)$ with $r_4 = 5$ and a URD$(\hat{K};\, 156)$ with $r_4 = 5$ exist by Lemma 2.6. □

Lemma 5.2. *There exists a URD*$(\hat{K};\, v)$ *with* $r_4 = 5$ *for all* $v \equiv 60 \pmod{72}$ *except possibly when* $v = 276, 348$.

Proof. A URD$(\hat{K};\, 36)$ with $r_4 = 5$ exists by Lemma 2.6 and therefore a URD$(\hat{K};\, 108)$ with $r_4 = 5$ exists by Theorem 3.3. Since $108 = v_0 = 9 \cdot r_4 + 6 \cdot j + 9$, with $j = 9$, then by Theorem 3.10 there exists a $\hat{K}$-frame of type $(24;\, 3^{12})^t (v_0 - 9;\, 3^{42}, 4^{r_4})^1$ for all $t \equiv 1 \pmod 3$ with $t \geq 1 + \frac{3\cdot(r_4+j)}{4} = 11.5$. Set $t = 10 + 3 \cdot i$ with $i = 1, 2, \ldots$. Therefore, by Theorem 3.9 there exist designs URD$(\hat{K};\, 24 \cdot (10 + 3 \cdot i) + 108)$ with $r_4 = 5$, $i = 1, 2, \ldots$. That is, all URD$(\hat{K};\, 348 + 72 \cdot i)$ with $r_4 = 5$, $i = 1, 2, \ldots$ exist.

The URD$(\hat{K};\, 60)$ with $r_4 = 5$ exists by Theorem 3.2. A URD$(\hat{K};\, 132)$ with $r_4 = 5$ and a URD$(\hat{K};\, 204)$ with $r_4 = 5$ exist by Lemma 2.6. □

Lemma 5.3. *There exists a URD*$(\hat{K};\, v)$ *with* $r_4 = 5$ *for all* $v \equiv 36 \pmod{72}$.

Proof. It follows from Lemma 2.6 and Theorem 3.5. □

Lemma 5.4. *There exists a URD*$(\hat{K};\, v)$ *with* $r_4 = 5$ *for* $v = 276$.

Proof. Take a 4-GDD of type $12^4 15^1$, which exists by Rees [5]. Apply Construction 3.11 with weight 4 and 3-frames of types 4^4, which is known to exist [7]. The result is a 3-frame of type $48^4 \cdot 60^1$. Take a 3-RGDD of type 24^3 and fill only two groups with URD$(\hat{K};\, 24)$ with $r_4 = 5, r_3 = 4$ from Lemma 2.6, that gives a so-called incomplete uniformly resolvable design (IURD). Adjoin 24 infinite points and fill all groups of size 48 of the frame with this IURD, whereas the infinite points form the hole. The 24 frame-3-parallel cases (pc) are completed with the 24 complete 3-pc of the IURD. Fill the group of size 60 together with the infinite points with a URD$(\hat{K};\, 84)$ with $r_4 = 5, r_3 = 34$ from Lemma 2.6, after completing the 30 frame-3-pc there remain 4 more 3-pc, which can be completed with the holey 3-pc of the IURDs. □

Lemma 5.5. *There exists a design URD*$(\hat{K};\, v)$ *with* $r_4 = 5$ *for* $v = 348$.

Proof. Take a 4-GDD of type $18^4 9^1$, which exists by Rees [5]. Apply Construction 3.11 with weight 4 and 3-frames of types 4^4, which is known to exist [7]. The result is a 3-frame of type $72^4 \cdot 36^1$. Take a 3-RGDD of type 24^4 and fill only three groups with URD($\hat{K}$; 24) with $r_4 = 5, r_3 = 4$ from Lemma 2.6, that gives a so-called incomplete uniformly resolvable design (IURD). Adjoin 24 infinite points and fill all groups of size 72 of the frame with this IURD, whereas the infinite points form the hole. Fill the group of size 36 together with the infinite points with a URD($\hat{K}$; 60) with $r_4 = 5, r_3 = 22$ from Theorem 3.2, which gives the design as required. □

All five lemmas of this section together with Lemma 2.6, Theorems 3.1 and 3.4 give our main result:

Theorem 5.6. *There exists a URD* $(\{3, 4\}; v)$ *with* $r_4 = 5$ *if, and only if,* $v \equiv 0 \pmod{12}$ *except when* $v = 12$.

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Appendix

This appendix contains ingredient designs required for our constructions. These were found computationally.

Example A.1. There exists a URD($\hat{K}$; 84) with $r_4 = 3$.

Proof. Let Z_λ be the group of residues modulo λ. The design is constructed on $X = Z_4 \times Z_{21}$. Take the following three parallel classes with blocks of size four:

$$P_1 = \{(0,0), (1,0), (2,0), (3,0)\} \ (\text{mod } (-, 21))$$
$$P_2 = \{(0,0), (1,1), (2,2), (3,3)\} \ (\text{mod } (-, 21))$$
$$P_3 = \{(0,3), (1,2), (2,1), (3,0)\} \ (\text{mod } (-, 21)).$$

It is well known that there is an RPBD(3; 21) with ten parallel classes. Place a copy of this design on each Z_{21} set. Denote the resolution classes by $R_{i,j}$ where $i \in Z_4$ denotes on which copy of Z_{21} the parallel class is placed and $j = 1, \dots, 10$ are the ten resolution classes. The parallel classes of the triples are formed as follows:

$\{(0,0),(1,4),(2,15)\}(\text{mod }(-,21)) \cup R_{3,1}$	$\{(0,0),(1,12),(3,4)\}(\text{mod }(-,21)) \cup R_{2,1}$
$\{(0,0),(1,10),(2,6)\}(\text{mod }(-,21)) \cup R_{3,2}$	$\{(0,0),(1,8),(3,13)\}(\text{mod }(-,21)) \cup R_{2,2}$
$\{(0,0),(1,19),(2,16)\}(\text{mod }(-,21)) \cup R_{3,3}$	$\{(0,0),(1,17),(3,5)\}(\text{mod }(-,21)) \cup R_{2,3}$
$\{(0,0),(1,2),(2,11)\}(\text{mod }(-,21)) \cup R_{3,4}$	$\{(0,0),(1,14),(3,7)\}(\text{mod }(-,21)) \cup R_{2,4}$
$\{(0,0),(1,13),(2,8)\}(\text{mod }(-,21)) \cup R_{3,5}$	$\{(0,0),(1,9),(3,6)\}(\text{mod }(-,21)) \cup R_{2,5}$
$\{(0,0),(1,6),(2,18)\}(\text{mod }(-,21)) \cup R_{3,6}$	$\{(0,0),(1,3),(3,10)\}(\text{mod }(-,21)) \cup R_{2,6}$
$\{(0,0),(1,18),(2,1)\}(\text{mod }(-,21)) \cup R_{3,7}$	$\{(0,0),(1,5),(3,17)\}(\text{mod }(-,21)) \cup R_{2,7}$
$\{(0,0),(1,7),(2,14)\}(\text{mod }(-,21)) \cup R_{3,8}$	$\{(0,0),(1,15),(3,14)\}(\text{mod }(-,21)) \cup R_{2,8}$
$\{(0,0),(1,16),(2,10)\}(\text{mod }(-,21)) \cup R_{3,9}$	$\{(0,0),(1,11),(3,12)\}(\text{mod }(-,21)) \cup R_{2,9}$

$\{(0,0),(2,7),(3,2)\}(\text{mod }(-,21)) \cup R_{1,1}$	$\{(1,0),(2,14),(3,4)\}(\text{mod }(-,21)) \cup R_{0,1}$
$\{(0,0),(2,17),(3,11)\}(\text{mod }(-,21)) \cup R_{1,2}$	$\{(1,0),(2,6),(3,3)\}(\text{mod }(-,21)) \cup R_{0,2}$
$\{(0,0),(2,3),(3,1)\}(\text{mod }(-,21)) \cup R_{1,3}$	$\{(1,0),(2,8),(3,11)\}(\text{mod }(-,21)) \cup R_{0,3}$
$\{(0,0),(2,12),(3,20)\}(\text{mod }(-,21)) \cup R_{1,4}$	$\{(1,0),(2,13),(3,15)\}(\text{mod }(-,21)) \cup R_{0,4}$
$\{(0,0),(2,5),(3,19)\}(\text{mod }(-,21)) \cup R_{1,5}$	$\{(1,0),(2,2),(3,6)\}(\text{mod }(-,21)) \cup R_{0,5}$
$\{(0,0),(2,9),(3,15)\}(\text{mod }(-,21)) \cup R_{1,6}$	$\{(1,0),(2,10),(3,17)\}(\text{mod }(-,21)) \cup R_{0,6}$
$\{(0,0),(2,13),(3,9)\}(\text{mod }(-,21)) \cup R_{1,7}$	$\{(1,0),(2,3),(3,16)\}(\text{mod }(-,21)) \cup R_{0,7}$
$\{(0,0),(2,20),(3,8)\}(\text{mod }(-,21)) \cup R_{1,8}$	$\{(1,0),(2,19),(3,8)\}(\text{mod }(-,21)) \cup R_{0,8}$
$\{(0,0),(2,4),(3,16)\}(\text{mod }(-,21)) \cup R_{1,9}$	$\{(1,0),(2,5),(3,10)\}(\text{mod }(-,21)) \cup R_{0,9}$

The last parallel class of triples is given by $\bigcup_{i=0}^{3} R_{i,10}$. □

Example A.2. There exists a URD($\hat{K}$; 156) with $r_4 = 3$.

Proof. Let Z_λ be the group of residues modulo λ. The design is constructed on $X = Z_4 \times Z_{39}$. Take the following three parallel classes with blocks of size four:

$$P_1 = \{(0,0), (1,0), (2,0), (3,0)\} \ (\text{mod } (-, 39))$$
$$P_2 = \{(0,0), (1,1), (2,2), (3,3)\} \ (\text{mod } (-, 39))$$
$$P_3 = \{(0,4), (1,3), (2,2), (3,0)\} \ (\text{mod } (-, 39)).$$

It is well known that there is an RPBD(3; 39) with 19 parallel classes. Place a copy of this design on each Z_{39} set. Denote the resolution classes by $R_{i,j}$ where $i \in Z_4$ denotes on which copy of Z_{39} the parallel class is placed and $j = 1, \ldots, 19$ are the resolution classes. The parallel classes of the triples are formed as follows:

$\{(0,0),(1,9),(2,26)\}(\text{mod }(-,39)) \cup R_{3,1}$	$\{(0,0),(1,10),(3,18)\}(\text{mod }(-,39)) \cup R_{2,1}$
$\{(0,0),(1,32),(2,5)\}(\text{mod }(-,39)) \cup R_{3,2}$	$\{(0,0),(1,29),(3,33)\}(\text{mod }(-,39)) \cup R_{2,2}$
$\{(0,0),(1,6),(2,4)\}(\text{mod }(-,39)) \cup R_{3,3}$	$\{(0,0),(1,25),(3,26)\}(\text{mod }(-,39)) \cup R_{2,3}$
$\{(0,0),(1,17),(2,33)\}(\text{mod }(-,39)) \cup R_{3,4}$	$\{(0,0),(1,36),(3,19)\}(\text{mod }(-,39)) \cup R_{2,4}$
$\{(0,0),(1,11),(2,8)\}(\text{mod }(-,39)) \cup R_{3,5}$	$\{(0,0),(1,2),(3,7)\}(\text{mod }(-,39)) \cup R_{2,5}$
$\{(0,0),(1,31),(2,3)\}(\text{mod }(-,39)) \cup R_{3,6}$	$\{(0,0),(1,5),(3,34)\}(\text{mod }(-,39)) \cup R_{2,6}$
$\{(0,0),(1,15),(2,23)\}(\text{mod }(-,39)) \cup R_{3,7}$	$\{(0,0),(1,12),(3,1)\}(\text{mod }(-,39)) \cup R_{2,7}$
$\{(0,0),(1,19),(2,29)\}(\text{mod }(-,39)) \cup R_{3,8}$	$\{(0,0),(1,3),(3,6)\}(\text{mod }(-,39)) \cup R_{2,8}$
$\{(0,0),(1,24),(2,9)\}(\text{mod }(-,39)) \cup R_{3,9}$	$\{(0,0),(1,27),(3,9)\}(\text{mod }(-,39)) \cup R_{2,9}$
$\{(0,0),(1,33),(2,36)\}(\text{mod }(-,39)) \cup R_{3,10}$	$\{(0,0),(1,20),(3,32)\}(\text{mod }(-,39)) \cup R_{2,10}$
$\{(0,0),(1,14),(2,28)\}(\text{mod }(-,39)) \cup R_{3,11}$	$\{(0,0),(1,23),(3,30)\}(\text{mod }(-,39)) \cup R_{2,11}$
$\{(0,0),(1,7),(2,20)\}(\text{mod }(-,39)) \cup R_{3,12}$	$\{(0,0),(1,13),(3,36)\}(\text{mod }(-,39)) \cup R_{2,12}$
$\{(0,0),(1,34),(2,24)\}(\text{mod }(-,39)) \cup R_{3,13}$	$\{(0,0),(1,18),(3,4)\}(\text{mod }(-,39)) \cup R_{2,13}$
$\{(0,0),(1,37),(2,18)\}(\text{mod }(-,39)) \cup R_{3,14}$	$\{(0,0),(1,28),(3,21)\}(\text{mod }(-,39)) \cup R_{2,14}$
$\{(0,0),(1,16),(2,21)\}(\text{mod }(-,39)) \cup R_{3,15}$	$\{(0,0),(1,4),(3,17)\}(\text{mod }(-,39)) \cup R_{2,15}$
$\{(0,0),(1,8),(2,14)\}(\text{mod }(-,39)) \cup R_{3,16}$	$\{(0,0),(1,35),(3,31)\}(\text{mod }(-,39)) \cup R_{2,16}$
$\{(0,0),(1,22),(2,17)\}(\text{mod }(-,39)) \cup R_{3,17}$	$\{(0,0),(1,26),(3,13)\}(\text{mod }(-,39)) \cup R_{2,17}$
$\{(0,0),(1,21),(2,15)\}(\text{mod }(-,39)) \cup R_{3,18}$	$\{(0,0),(1,30),(3,11)\}(\text{mod }(-,39)) \cup R_{2,18}$

$\{(0,0),(2,12),(3,37)\}(\text{mod }(-,39)) \cup R_{1,1}$	$\{(1,0),(2,7),(3,24)\}(\text{mod }(-,39)) \cup R_{0,1}$
$\{(0,0),(2,34),(3,38)\}(\text{mod }(-,39)) \cup R_{1,2}$	$\{(1,0),(2,32),(3,16)\}(\text{mod }(-,39)) \cup R_{0,2}$
$\{(0,0),(2,30),(3,15)\}(\text{mod }(-,39)) \cup R_{1,3}$	$\{(1,0),(2,25),(3,38)\}(\text{mod }(-,39)) \cup R_{0,3}$
$\{(0,0),(2,35),(3,28)\}(\text{mod }(-,39)) \cup R_{1,4}$	$\{(1,0),(2,15),(3,17)\}(\text{mod }(-,39)) \cup R_{0,4}$
$\{(0,0),(2,38),(3,10)\}(\text{mod }(-,39)) \cup R_{1,5}$	$\{(1,0),(2,9),(3,14)\}(\text{mod }(-,39)) \cup R_{0,5}$
$\{(0,0),(2,6),(3,20)\}(\text{mod }(-,39)) \cup R_{1,6}$	$\{(1,0),(2,28),(3,10)\}(\text{mod }(-,39)) \cup R_{0,6}$
$\{(0,0),(2,16),(3,24)\}(\text{mod }(-,39)) \cup R_{1,7}$	$\{(1,0),(2,35),(3,15)\}(\text{mod }(-,39)) \cup R_{0,7}$
$\{(0,0),(2,13),(3,12)\}(\text{mod }(-,39)) \cup R_{1,8}$	$\{(1,0),(2,18),(3,9)\}(\text{mod }(-,39)) \cup R_{0,8}$
$\{(0,0),(2,27),(3,8)\}(\text{mod }(-,39)) \cup R_{1,9}$	$\{(1,0),(2,31),(3,19)\}(\text{mod }(-,39)) \cup R_{0,9}$
$\{(0,0),(2,7),(3,16)\}(\text{mod }(-,39)) \cup R_{1,10}$	$\{(1,0),(2,4),(3,30)\}(\text{mod }(-,39)) \cup R_{0,10}$
$\{(0,0),(2,25),(3,2)\}(\text{mod }(-,39)) \cup R_{1,11}$	$\{(1,0),(2,2),(3,31)\}(\text{mod }(-,39)) \cup R_{0,11}$
$\{(0,0),(2,19),(3,14)\}(\text{mod }(-,39)) \cup R_{1,12}$	$\{(1,0),(2,19),(3,34)\}(\text{mod }(-,39)) \cup R_{0,12}$
$\{(0,0),(2,31),(3,27)\}(\text{mod }(-,39)) \cup R_{1,13}$	$\{(1,0),(2,30),(3,37)\}(\text{mod }(-,39)) \cup R_{0,13}$
$\{(0,0),(2,32),(3,29)\}(\text{mod }(-,39)) \cup R_{1,14}$	$\{(1,0),(2,22),(3,11)\}(\text{mod }(-,39)) \cup R_{0,14}$
$\{(0,0),(2,1),(3,23)\}(\text{mod }(-,39)) \cup R_{1,15}$	$\{(1,0),(2,26),(3,18)\}(\text{mod }(-,39)) \cup R_{0,15}$
$\{(0,0),(2,11),(3,5)\}(\text{mod }(-,39)) \cup R_{1,16}$	$\{(1,0),(2,23),(3,33)\}(\text{mod }(-,39)) \cup R_{0,16}$
$\{(0,0),(2,22),(3,25)\}(\text{mod }(-,39)) \cup R_{1,17}$	$\{(1,0),(2,27),(3,6)\}(\text{mod }(-,39)) \cup R_{0,17}$
$\{(0,0),(2,10),(3,22)\}(\text{mod }(-,39)) \cup R_{1,18}$	$\{(1,0),(2,21),(3,27)\}(\text{mod }(-,39)) \cup R_{0,18}$

The last parallel class of triples is given by $\bigcup_{i=0}^{3} R_{i,19}$.

Unless otherwise stated, the element set of a design with v points is labeled $1, 2, \ldots, v$. □

Example A.3. An $\text{LURD}_4(\hat{K}; 12)$ with $r_4 = 2$; each row forms a parallel class:

(1 8 12; 0 3 3), (2 3 11; 0 0 0), (4 6 10; 1 0 3), (5 7 9; 2 1 3),
(8 9 10; 2 0 2), (2 6 11; 3 1 2), (1 4 5; 0 1 1), (3 7 12; 3 1 2),
(2 7 9; 0 2 2), (5 10 12; 1 1 0), (3 4 8; 0 0 0), (1 6 11; 0 1 1),
(5 6 9; 3 2 3), (4 7 8; 2 1 3), (2 3 10; 2 3 1), (1 11 12; 2 1 3),
(1 7 9; 0 1 1), (3 6 10; 3 0 1), (4 5 12; 2 2 0), (2 8 11; 0 2 2),
(6 9 10; 0 0 0), (2 4 7; 2 3 1), (1 3 8; 2 3 1), (5 11 12; 0 2 2),
(2 7 12; 2 2 0), (1 5 8; 2 2 0), (3 6 9; 1 3 2), (4 10 11; 2 1 3),
(7 10 11; 1 1 0), (6 8 12; 2 2 0), (3 4 5; 1 0 3), (1 2 9; 2 3 1),
(3 5 9; 3 2 3), (1 4 7; 2 1 3), (2 10 12; 1 3 2), (6 8 11; 0 3 3),
(5 8 10; 1 2 1), (1 2 11; 0 3 3), (4 6 7; 0 0 0), (3 9 12; 1 0 3),
(2 4 6; 3 2 3), (1 7 9; 2 2 0), (8 10 12; 2 1 3), (3 5 11; 2 1 3),
(2 5 8; 2 1 3), (3 7 11; 0 3 3), (9 10 12; 1 2 1), (1 4 6; 3 1 2),
(2 9 12; 3 0 1), (6 7 8; 3 3 0), (1 3 10; 0 2 2), (4 5 11; 0 2 2),
(3 5 11; 1 2 1), (2 4 9; 0 0 0), (1 6 12; 2 2 0), (7 8 10; 1 0 3),

(2 3 6; 3 1 2), (4 8 12; 3 1 2), (7 10 11; 2 0 2), (1 5 9; 0 0 0),
(9 11 12; 0 0 0), (5 6 7; 2 3 1), (1 4 10; 1 0 3), (2 3 8; 1 3 2),
(2 4 12; 1 1 0), (5 7 10; 1 0 3), (8 9 11; 1 0 3), (1 3 6; 3 3 0),
(2 5 7; 1 1 0), (6 8 9; 1 1 0), (3 4 12; 3 2 3), (1 10 11; 3 0 1),
(4 9 11; 3 0 1), (3 7 8; 1 3 2), (1 2 10; 1 1 0), (5 6 12; 0 3 3),
(6 7 11 12; 2 0 1 2 3 1), (1 2 5 8; 3 3 1 0 2 2), (3 4 9 10; 2 0 3 2 1 3),
(4 8 9 11; 2 1 3 3 1 2), (2 5 6 10; 3 0 2 1 3 2), (1 3 7 12; 1 3 0 2 3 1).

Example A.4. $\mathrm{LURD}_3(\hat{K}; 24)$ with $r_4 = 3$; each row forms a parallel class:

(1 17 23; 1 0 2), (2 18 24; 1 2 1), (3 19 21; 0 1 1), (4 20 22; 0 0 0), (5 9 13; 2 0 1), (6 10 14; 0 2 2), (7 11 15; 2 2 0), (8 12 16; 2 1 2),
(5 11 21; 0 1 1), (6 12 22; 0 2 2), (7 9 23; 0 1 1), (8 10 24; 0 1 1), (1 16 18; 0 0 0), (2 13 19; 0 0 0), (3 14 20; 0 1 1), (4 15 17; 0 1 1),
(5 14 17; 1 0 2), (6 15 18; 2 1 2), (7 16 19; 0 1 1), (8 13 20; 1 1 0), (1 9 21; 1 2 1), (2 10 22; 1 1 0), (3 11 23; 1 0 2), (4 12 24; 0 2 2),
(9 14 18; 2 1 2), (10 15 19; 0 1 1), (11 16 20; 0 0 0), (12 13 17; 2 2 0), (1 5 22; 0 0 0), (2 6 23; 2 1 2), (3 7 24; 0 2 2), (4 8 21; 2 1 2),
(1 10 20; 0 1 1), (2 11 17; 0 0 0), (3 12 18; 1 1 0), (4 9 19; 1 2 1), (5 16 23; 0 1 1), (6 13 24; 1 0 2), (7 14 21; 1 2 1), (8 15 22; 1 0 2),
(1 15 24; 1 0 2), (2 16 21; 0 0 0), (3 13 22; 0 2 2), (4 14 23; 2 0 1), (5 12 20; 1 1 0), (6 9 17; 2 0 1), (7 10 18; 1 0 2), (8 11 19; 2 2 0),
(13 18 21; 0 1 1), (14 19 22; 2 1 2), (15 20 23; 1 0 2), (16 17 24; 1 0 2), (1 7 12; 0 1 1), (2 8 9; 1 1 0), (3 5 10; 0 2 2), (4 6 11; 2 2 0),
(1 6 19; 0 1 1), (2 7 20; 2 2 0), (3 8 17; 2 2 0), (4 5 18; 1 1 0), (9 16 22; 0 0 0), (10 13 23; 0 0 0), (11 14 24; 2 1 2), (12 15 21; 0 1 1),
(9 20 24; 0 2 2), (10 17 21; 2 1 2), (11 18 22; 0 0 0), (12 19 23; 1 2 1), (1 8 14; 2 2 0), (2 5 15; 1 0 2), (3 6 16; 2 0 1), (4 7 13; 1 2 1),
(5 19 24; 0 1 1), (6 20 21; 0 0 0), (7 17 22; 0 1 1), (8 18 23; 0 0 0), (1 11 13; 1 0 2), (2 12 14; 2 0 1), (3 9 15; 2 2 0), (4 10 16; 1 2 1),
(2 18 24; 2 1 2), (1 3 19; 2 0 1), (4 20 22; 1 2 1), (5 21 23; 0 2 2), (6 10 14; 1 1 0), (7 11 15; 1 0 2), (8 12 16; 0 0 0), (9 13 17; 0 2 2),
(6 12 22; 2 0 1), (7 13 23; 2 0 1), (8 10 24; 2 2 0), (1 9 11; 0 2 2), (2 17 19; 2 1 2), (3 14 20; 2 2 0), (4 15 21; 2 2 0), (5 16 18; 1 2 1),
(6 15 18; 1 2 1), (7 16 19; 2 2 0), (8 17 20; 1 0 2), (9 14 21; 1 0 2), (2 10 22; 2 0 1), (3 11 23; 2 2 0), (4 12 24; 2 0 1), (1 5 13; 2 1 2),
(10 15 19; 1 0 2), (11 16 20; 2 1 2), (12 17 21; 0 0 0), (13 14 18; 1 2 1), (2 6 23; 0 0 0), (3 7 24; 2 0 1), (1 4 8; 1 1 0), (5 9 22; 1 2 1),
(2 11 21; 2 1 2), (3 12 18; 2 0 1), (4 13 19; 0 1 1), (5 10 20; 1 0 2), (6 17 24; 2 2 0), (1 7 14; 2 1 2), (8 15 22; 0 1 1), (9 16 23; 2 2 0),
(1 2 16; 0 1 1), (3 17 22; 1 1 0), (4 14 23; 0 2 2), (5 15 24; 1 2 1), (6 13 21; 2 1 2), (7 10 18; 2 2 0), (8 11 19; 1 0 2), (9 12 20; 1 2 1),
(14 19 22; 0 0 0), (15 20 23; 2 2 0), (16 21 24; 2 1 2), (1 17 18; 0 1 1), (2 8 13; 2 2 0), (3 9 10; 0 0 0), (4 6 11; 0 1 1), (5 7 12; 1 0 2),
(2 7 20; 0 1 1), (3 8 21; 1 2 1), (4 9 18; 0 0 0), (5 6 19; 1 1 0), (10 17 23; 0 1 1), (11 14 24; 0 0 0), (1 12 15; 0 2 2), (13 16 22; 0 1 1),
(1 10 21; 1 1 0), (11 18 22; 1 2 1), (12 19 23; 0 0 0), (13 20 24; 1 1 0), (2 9 15; 0 1 1), (3 6 16; 0 2 2), (4 7 17; 2 0 1), (5 8 14; 1 2 1),
(1 6 20; 1 0 2), (7 21 22; 0 0 0), (8 18 23; 1 2 1), (9 19 24; 0 0 0), (2 12 14; 0 2 2), (3 13 15; 2 0 1), (4 10 16; 2 1 2), (5 11 17; 1 2 1),
(1 3 19; 0 2 2), (2 4 20; 1 0 2), (5 21 23; 2 0 1), (6 22 24; 1 1 0), (7 11 15; 0 1 1), (8 12 16; 1 2 1), (9 13 17; 2 0 1), (10 14 18; 1 1 0),
(7 13 23; 0 2 2), (8 14 24; 2 0 1), (1 9 11; 2 0 1), (2 10 12; 0 1 1), (3 18 20; 2 0 1), (4 15 21; 1 0 2), (5 16 22; 2 1 2), (6 17 19; 1 2 1),
(7 16 19; 1 0 2), (8 17 20; 2 2 0), (9 18 21; 2 2 0), (10 15 22; 2 2 0), (3 11 23; 0 1 1), (4 12 24; 1 1 0), (1 5 13; 1 2 1), (2 6 14; 1 1 0),
(11 16 20; 1 2 1), (12 17 21; 1 2 1), (13 18 22; 1 0 2), (14 15 19; 1 1 0), (3 7 24; 1 1 0), (1 4 8; 2 0 1), (2 5 9; 2 2 0), (6 10 23; 2 1 2),
(3 12 22; 0 0 0), (4 13 19; 1 0 2), (5 14 20; 0 2 2), (6 11 21; 2 2 0), (1 7 18; 1 2 1), (2 8 15; 0 2 2), (9 16 23; 1 0 2), (10 17 24; 1 2 1),
(2 3 17; 1 1 0), (4 18 23; 2 1 2), (5 15 24; 0 0 0), (1 6 16; 2 2 0), (7 14 22; 0 2 2), (8 11 19; 0 1 1), (9 12 20; 2 1 2), (10 13 21; 2 2 0),
(15 20 23; 0 1 1), (16 21 24; 1 2 1), (1 17 22; 2 1 2), (2 18 19; 0 2 2), (3 9 14; 1 1 0), (4 10 11; 0 0 0), (5 7 12; 2 2 0), (6 8 13; 1 0 2),
(3 8 21; 0 0 0), (4 9 22; 2 1 2), (5 10 19; 0 2 2), (6 7 20; 2 1 2), (11 18 24; 2 2 0), (1 12 15; 2 0 1), (2 13 16; 1 2 1), (14 17 23; 0 0 0),
(2 11 22; 1 2 1), (12 19 23; 2 1 2), (13 20 24; 2 0 1), (1 14 21; 0 0 0), (3 10 16; 1 1 0), (4 7 17; 0 2 2), (5 8 18; 2 1 2), (6 9 15; 1 0 2),
(2 7 21; 1 2 1), (8 22 23; 2 1 2), (9 19 24; 2 1 2), (1 10 20; 2 2 0), (3 13 15; 1 1 0), (4 14 16; 1 0 2), (5 11 17; 2 1 2), (6 12 18; 1 0 2),
(1 2 3 4; 1 1 0 0 2 2), (5 6 7 8; 0 0 0 0 0 0), (9 10 11 12; 1 0 0 2 2 0), (13 14 15 16; 2 2 2 0 0 0), (17 18 19 20; 2 0 1 1 2 1), (21 22 23 24; 2 0 0 1 1 0),
(2 3 4 5; 2 0 0 1 1 0), (6 7 8 9; 1 2 0 1 2 1), (10 11 12 13; 1 0 1 2 0 1), (14 15 16 17; 2 1 1 2 2 0), (18 19 20 21; 0 0 2 0 2 2), (1 22 23 24; 2 2 1 0 2 2),
(3 4 5 6; 0 2 1 2 1 2), (7 8 9 10; 2 1 0 2 1 2), (11 12 13 14; 1 1 1 0 0 0), (15 16 17 18; 1 0 0 2 2 0), (19 20 21 22; 2 0 1 1 2 1), (1 2 23 24; 2 1 2 2 0 1).

Example A.5. An $\mathrm{LURD}_4(\hat{K}; 24)$ with $r_4 = 2$; each row forms a parallel class:

(1 9 24; 3 0 1), (7 19 20; 0 1 1), (2 14 23; 0 1 1), (3 6 13; 2 2 0), (12 17 21; 0 3 3), (8 10 16; 1 2 1), (4 5 22; 0 2 2), (11 15 18; 1 2 1),
(2 10 24; 3 0 1), (8 20 21; 3 2 3), (1 3 15; 2 1 3), (4 7 14; 3 2 3), (13 18 22; 3 1 2), (9 11 17; 3 0 1), (5 6 23; 3 0 1), (12 16 19; 2 3 1),
(3 11 24; 0 0 0), (9 21 22; 1 0 3), (2 4 16; 0 1 1), (5 8 15; 0 1 1), (14 19 23; 0 0 0), (10 12 18; 0 0 0), (1 6 7; 0 0 0), (13 17 20; 2 1 3),
(4 12 24; 2 2 0), (10 22 23; 3 3 0), (3 5 17; 3 1 2), (6 9 16; 1 0 3), (1 15 20; 0 0 0), (11 13 19; 3 2 3), (2 7 8; 0 0 0), (14 18 21; 0 1 1),
(5 13 24; 3 1 2), (1 11 23; 0 0 0), (4 6 18; 0 1 1), (7 10 17; 1 1 0), (2 16 21; 0 2 2), (12 14 20; 3 2 3), (3 8 9; 2 2 0), (15 19 22; 1 1 0),
(6 14 24; 2 2 0), (1 2 12; 2 2 0), (5 7 19; 1 0 3), (8 11 18; 1 1 0), (3 17 22; 0 0 0), (13 15 21; 1 2 1), (4 9 10; 1 2 1), (16 20 23; 2 3 1),
(7 15 24; 0 2 2), (2 3 13; 2 2 0), (6 8 20; 3 3 0), (9 12 19; 0 1 1), (4 18 23; 0 0 0), (14 16 22; 1 2 1), (5 10 11; 2 2 0), (1 17 21; 1 3 2),
(8 16 24; 3 1 2), (3 4 14; 1 2 1), (7 9 21; 2 0 2), (10 13 20; 1 1 0), (1 5 19; 2 1 3), (15 17 23; 1 2 1), (6 11 12; 0 0 0), (2 18 22; 0 0 0),
(9 17 24; 2 3 1), (4 5 15; 3 1 2), (8 10 22; 0 1 1), (11 14 21; 2 2 0), (2 6 20; 2 0 2), (1 16 18; 0 1 1), (7 12 13; 3 3 0), (3 19 23; 0 1 1),
(10 18 24; 1 2 1), (5 6 16; 0 3 3), (9 11 23; 2 3 1), (12 15 22; 3 1 2), (3 7 21; 2 3 1), (2 17 19; 0 2 2), (8 13 14; 2 2 0), (1 4 20; 0 2 2),
(11 19 24; 1 3 2), (6 7 17; 3 1 2), (1 10 12; 1 0 3), (13 16 23; 2 2 0), (4 8 22; 0 3 3), (3 18 20; 3 3 0), (9 14 15; 3 0 1), (2 5 21; 2 1 3),
(12 20 24; 3 3 0), (7 8 18; 3 2 3), (2 11 13; 0 0 0), (1 14 17; 3 0 1), (5 9 23; 3 3 0), (4 19 21; 0 1 1), (10 15 16; 3 0 1), (3 6 22; 1 1 0),
(13 21 24; 3 3 0), (8 9 19; 1 1 0), (3 12 14; 1 1 0), (2 15 18; 2 1 3), (1 6 10; 3 0 1), (5 20 22; 3 3 0), (11 16 17; 2 0 2), (4 7 23; 0 1 1),
(14 22 24; 3 3 0), (9 10 20; 0 2 2), (4 13 15; 0 2 2), (3 16 19; 0 2 2), (2 7 11; 2 2 0), (6 21 23; 0 3 3), (12 17 18; 2 1 3), (1 5 8; 0 1 1),
(15 23 24; 3 1 2), (10 11 21; 3 2 3), (5 14 16; 3 2 3), (4 17 20; 1 3 2), (3 8 12; 1 0 3), (1 7 22; 3 2 3), (13 18 19; 0 0 0), (2 6 9; 3 2 3),
(1 16 24; 1 1 0), (11 12 22; 1 3 2), (6 15 17; 2 0 2), (5 18 21; 0 2 2), (4 9 13; 0 3 3), (2 8 23; 2 2 0), (14 19 20; 1 0 3), (3 7 10; 0 0 0),
(2 17 24; 1 1 0), (12 13 23; 2 1 3), (7 16 18; 2 1 3), (6 19 22; 0 1 1), (5 10 14; 3 0 1), (1 3 9; 0 0 0), (15 20 21; 2 2 0), (4 8 11; 3 2 3),
(3 18 24; 1 3 2), (1 13 14; 0 2 2), (8 17 19; 0 0 0), (7 20 23; 2 2 0), (6 11 15; 1 3 2), (2 4 10; 3 0 1), (16 21 22; 1 2 1), (5 9 12; 0 2 2),
(4 19 24; 1 0 3), (2 14 15; 1 0 3), (9 18 20; 2 0 2), (1 8 21; 3 2 3), (7 12 16; 0 0 0), (3 5 11; 0 1 1), (17 22 23; 2 0 2), (6 10 13; 2 1 3),
(5 20 24; 1 0 3), (3 15 16; 0 3 3), (10 19 21; 2 1 3), (2 9 22; 0 3 3), (8 13 17; 3 2 3), (4 6 12; 2 3 1), (1 18 23; 2 1 3), (7 11 14; 2 1 3),
(6 21 24; 1 0 3), (4 16 17; 3 2 3), (11 20 22; 2 1 3), (3 10 23; 3 3 0), (9 14 18; 2 0 2), (5 7 13; 3 0 1), (1 2 19; 0 3 3), (8 12 15; 2 2 0),
(7 22 24; 0 1 1), (5 17 18; 3 3 0), (12 21 23; 2 2 0), (1 4 11; 2 3 1), (10 15 19; 0 0 0), (6 8 14; 2 1 3), (2 3 20; 3 1 2), (9 13 16; 2 2 0),
(8 23 24; 3 0 1), (6 18 19; 3 1 2), (1 13 22; 1 3 2), (2 5 12; 3 2 3), (11 16 20; 0 1 1), (7 9 15; 1 3 2), (3 4 21; 3 2 3), (10 14 17; 3 1 2),
(1 17 23; 3 2 3), (2 18 24; 3 2 3), (3 19 21; 1 1 0), (4 20 22; 0 1 1), (5 9 13; 1 2 1), (6 10 14; 0 0 0), (7 11 15; 3 2 3), (8 12 16; 1 0 3),
(5 11 21; 0 1 1), (6 12 22; 3 2 3), (7 9 23; 3 0 1), (8 10 24; 3 2 3), (1 16 18; 3 3 0), (2 13 19; 3 0 1), (3 14 20; 3 0 1), (4 15 17; 3 3 0),
(5 14 17; 1 0 3), (6 15 18; 0 2 2), (7 16 19; 3 2 3), (8 13 20; 0 2 2), (1 9 21; 2 1 3), (2 10 22; 2 2 0), (3 11 23; 3 2 3), (4 12 24; 0 1 1),

(9 14 18; 0 3 3), (10 15 19; 1 3 2), (11 16 20; 3 3 0), (12 13 17; 1 1 0), (1 5 22; 3 0 1), (2 6 23; 1 3 2), (3 7 24; 3 2 3), (4 8 21; 2 2 0),
(1 10 20; 2 1 3), (2 11 17; 3 2 3), (3 12 18; 3 2 3), (4 9 19; 3 2 3), (5 16 23; 0 1 1), (6 13 24; 3 3 0), (7 14 21; 0 2 2), (8 15 22; 3 2 3),
(1 15 24; 3 3 0), (2 16 21; 3 3 0), (3 13 22; 3 3 0), (4 14 23; 3 2 3), (5 12 20; 1 2 1), (6 9 17; 0 3 3), (7 10 18; 2 0 2), (8 11 19; 0 3 3),
(13 18 21; 1 1 0), (14 19 22; 2 1 3), (15 20 23; 3 1 2), (16 17 24; 0 3 3), (1 7 12; 2 3 1), (2 8 9; 3 1 2), (3 5 10; 1 1 0), (4 6 11; 1 3 2),
(1 6 19; 2 0 2), (2 7 20; 3 3 0), (3 8 17; 0 3 3), (4 5 18; 1 3 2), (9 16 22; 1 1 0), (10 13 23; 2 2 0), (11 14 24; 1 2 1), (12 15 21; 2 1 3),
(9 20 24; 3 0 1), (10 17 21; 3 3 0), (11 18 22; 3 2 3), (12 19 23; 0 3 3), (1 8 14; 0 0 0), (2 5 15; 0 3 3), (3 6 16; 0 2 2), (4 7 13; 2 2 0),
(5 19 24; 1 2 1), (6 20 21; 1 2 1), (7 17 22; 0 1 1), (8 18 23; 0 2 2), (1 11 13; 1 3 2), (2 12 14; 1 2 1), (3 9 15; 1 2 1), (4 10 16; 0 2 2),
(1 3 19; 3 2 3), (2 4 20; 1 2 1), (5 21 23; 0 2 2), (6 22 24; 3 1 2), (7 11 15; 1 1 0), (8 12 16; 0 1 1), (9 13 17; 0 1 1), (10 14 18; 2 3 1),
(7 13 23; 2 3 1), (8 14 24; 1 3 2), (1 9 11; 1 2 1), (2 10 12; 1 3 2), (3 18 20; 0 1 1), (4 15 21; 0 0 0), (5 16 22; 1 0 3), (6 17 19; 2 3 1),
(7 16 19; 1 1 0), (8 17 20; 1 1 0), (9 18 21; 1 0 3), (10 15 22; 2 2 0), (3 11 23; 2 0 2), (4 12 24; 1 3 2), (1 5 13; 1 2 1), (2 6 14; 0 3 3),
(11 16 20; 1 0 3), (12 17 21; 3 0 1), (13 18 22; 2 3 1), (14 15 19; 0 3 3), (3 7 24; 1 1 0), (1 4 8; 1 2 1), (2 5 9; 1 3 2), (6 10 23; 3 0 1),
(3 12 22; 2 2 0), (4 13 19; 1 3 2), (5 14 20; 2 0 2), (6 11 21; 3 3 0), (1 7 18; 1 0 3), (2 8 15; 1 1 0), (9 16 23; 0 2 2), (10 17 24; 2 0 2),
(2 3 17; 1 3 2), (4 18 23; 2 3 1), (5 15 24; 0 3 3), (1 6 16; 1 2 1), (7 14 22; 2 2 0), (8 11 19; 2 2 0), (9 12 20; 1 1 0), (10 13 21; 0 0 0),
(15 20 23; 1 0 3), (16 21 24; 3 1 2), (1 17 22; 2 1 3), (2 18 19; 2 1 3), (3 9 14; 3 0 1), (4 10 11; 3 0 1), (5 7 12; 2 0 2), (6 8 13; 1 2 1),
(3 8 21; 3 0 1), (4 9 22; 2 0 2), (5 10 19; 1 2 1), (6 7 20; 1 0 3), (11 18 24; 1 1 0), (1 12 15; 1 2 1), (2 13 16; 1 2 1), (14 17 23; 0 2 2),
(2 11 22; 1 1 0), (12 19 23; 2 0 2), (13 20 24; 3 1 2), (1 14 21; 1 0 3), (3 10 16; 2 1 3), (4 7 17; 1 0 3), (5 8 18; 3 1 2), (6 9 15; 2 1 3),
(2 7 21; 1 0 3), (8 22 23; 0 1 1), (9 19 24; 2 2 0), (1 10 20; 3 3 0), (3 13 15; 1 1 0), (4 14 16; 0 0 0), (5 11 17; 3 1 2), (6 12 18; 2 0 2),
(1 2 3 4; 1 1 3 0 2 2), (5 6 7 8; 2 0 2 2 0 2), (9 10 11 12; 2 0 3 2 1 3), (13 14 15 16; 1 3 3 2 2 0), (17 18 19 20; 2 3 1 1 3 2), (21 22 23 24; 2 1 1 3 3 0),
(3 4 5 6; 0 2 3 2 3 1), (7 8 9 10; 1 0 3 3 2 3), (11 12 13 14; 2 1 0 3 2 3), (15 16 17 18; 2 3 0 1 2 1), (19 20 21 22; 0 2 2 2 2 0), (1 2 23 24; 3 3 2 0 3 3).

Example A.6. A URD($\hat{K}$; 24) with $r_4 = 3$ calculated with DESIGN [12]; each row forms a parallel class:

{1, 3, 6}, {2, 9, 11}, {4, 13, 24}, {5, 7, 10}, {8, 16, 17}, {12, 20, 21}, {14, 18, 22}, {15, 19, 23},
{1, 4, 11}, {2, 22, 23}, {3, 5, 8}, {6, 14, 15}, {7, 9, 12}, {10, 18, 19}, {13, 17, 21}, {16, 20, 24},
{1, 5, 9}, {2, 6, 10}, {3, 23, 24}, {4, 14, 21}, {7, 15, 16}, {8, 13, 18}, {11, 19, 20}, {12, 17, 22},
{1, 8, 10}, {2, 4, 7}, {3, 13, 20}, {5, 21, 23}, {6, 22, 24}, {9, 14, 19}, {11, 15, 17}, {12, 16, 18},
{1, 17, 19}, {2, 18, 20}, {3, 10, 12}, {4, 6, 9}, {5, 15, 22}, {7, 13, 23}, {8, 14, 24}, {11, 16, 21},
{1, 18, 23}, {2, 5, 12}, {3, 19, 21}, {4, 20, 22}, {6, 8, 11}, {7, 17, 24}, {9, 13, 15}, {10, 14, 16},
{1, 21, 22}, {2, 19, 24}, {3, 7, 11}, {4, 8, 12}, {5, 13, 14}, {6, 16, 23}, {9, 17, 18}, {10, 15, 20},
{1, 2, 13, 16}, {3, 4, 15, 18}, {5, 6, 17, 20}, {7, 8, 19, 22}, {9, 10, 21, 24}, {11, 12, 14, 23},
{1, 7, 14, 20}, {2, 8, 15, 21}, {3, 9, 16, 22}, {4, 10, 17, 23}, {5, 11, 18, 24}, {6, 12, 13, 19},
{1, 12, 15, 24}, {2, 3, 14, 17}, {4, 5, 16, 19}, {6, 7, 18, 21}, {8, 9, 20, 23}, {10, 11, 13, 22}.

Example A.7. A URD($\hat{K}$; 24) with $r_4 = 5$ calculated with DESIGN [12]; each row forms a parallel class:

{1, 5, 9}, {2, 6, 10}, {3, 7, 11}, {4, 8, 12}, {13, 17, 21}, {14, 18, 22}, {15, 19, 23}, {16, 20, 24},
{1, 6, 16}, {2, 15, 20}, {3, 10, 13}, {4, 9, 19}, {5, 18, 23}, {7, 12, 22}, {8, 14, 21}, {11, 17, 24},
{1, 8, 23}, {2, 7, 17}, {3, 16, 21}, {4, 11, 14}, {5, 10, 20}, {6, 19, 24}, {9, 15, 22}, {12, 13, 18},
{1, 14, 19}, {2, 9, 24}, {3, 8, 18}, {4, 17, 22}, {5, 12, 15}, {6, 11, 21}, {7, 13, 20}, {10, 16, 23},
{1, 2, 4, 13}, {3, 9, 17, 23}, {5, 21, 22, 24}, {6, 12, 14, 20}, {7, 8, 10, 19}, {11, 15, 16, 18},
{1, 3, 12, 24}, {2, 18, 19, 21}, {4, 5, 7, 16}, {6, 13, 22, 23}, {8, 9, 11, 20}, {10, 14, 15, 17},
{1, 7, 15, 21}, {2, 11, 12, 23}, {3, 19, 20, 22}, {4, 10, 18, 24}, {5, 6, 8, 17}, {9, 13, 14, 16},
{1, 10, 11, 22}, {2, 3, 5, 14}, {4, 20, 21, 23}, {6, 7, 9, 18}, {8, 13, 15, 24}, {12, 16, 17, 19},
{1, 17, 18, 20}, {2, 8, 16, 22}, {3, 4, 6, 15}, {5, 11, 13, 19}, {7, 14, 23, 24}, {9, 10, 12, 21}.

Example A.8. There exists a URD($\hat{K}$; 84) with $r_4 = 5$.

Proof. Let Z_λ be the group of residues modulo λ. The design is constructed on $X = Z_4 \times Z_{21}$. Take the following five parallel classes with blocks of size four:

$$P_1 = \{(0, 0), (1, 0), (2, 0), (3, 0)\} \pmod{(-, 21)}$$
$$P_2 = \{(0, 0), (1, 1), (2, 2), (3, 3)\} \pmod{(-, 21)}$$
$$P_3 = \{(0, 3), (1, 2), (2, 1), (3, 0)\} \pmod{(-, 21)}$$
$$P_4 = \{(0, 0), (1, 2), (2, 4), (3, 7)\} \pmod{(-, 21)}$$
$$P_5 = \{(0, 7), (1, 5), (2, 3), (3, 0)\} \pmod{(-, 21)}.$$

It is well known that there is an RPBD(3; 21) with ten parallel classes. Place a copy of this design on each Z_{21} set. Denote the resolution classes by $R_{i,j}$ where $i \in Z_4$ denotes on which copy of Z_{21} the parallel class is placed and $j = 1, \ldots, 10$ are the ten resolution classes. The parallel classes of the triples are formed as follows:

$\{(0, 0), (1, 11), (2, 1)\} \pmod{(-, 21)} \cup R_{3,1}$	$\{(0, 0), (2, 9), (3, 11)\} \pmod{(-, 21)} \cup R_{1,1}$
$\{(0, 0), (1, 7), (2, 3)\} \pmod{(-, 21)} \cup R_{3,2}$	$\{(0, 0), (2, 7), (3, 12)\} \pmod{(-, 21)} \cup R_{1,2}$
$\{(0, 0), (1, 16), (2, 20)\} \pmod{(-, 21)} \cup R_{3,3}$	$\{(0, 0), (2, 11), (3, 20)\} \pmod{(-, 21)} \cup R_{1,3}$
$\{(0, 0), (1, 10), (2, 5)\} \pmod{(-, 21)} \cup R_{3,4}$	$\{(0, 0), (2, 6), (3, 4)\} \pmod{(-, 21)} \cup R_{1,4}$
$\{(0, 0), (1, 13), (2, 10)\} \pmod{(-, 21)} \cup R_{3,5}$	$\{(0, 0), (2, 18), (3, 8)\} \pmod{(-, 21)} \cup R_{1,5}$
$\{(0, 0), (1, 15), (2, 8)\} \pmod{(-, 21)} \cup R_{3,6}$	$\{(0, 0), (2, 13), (3, 19)\} \pmod{(-, 21)} \cup R_{1,6}$

$\{(0,0),(1,4),(2,16)\}(\text{mod }(-,21))\cup R_{3,7}$	$\{(0,0),(2,14),(3,10)\}(\text{mod }(-,21))\cup R_{1,7}$
$\{(0,0),(1,5),(2,12)\}(\text{mod }(-,21))\cup R_{3,8}$	$\{(0,0),(2,15),(3,6)\}(\text{mod }(-,21))\cup R_{1,8}$
$\{(0,0),(1,6),(3,17)\}(\text{mod }(-,21))\cup R_{2,1}$	$\{(1,0),(2,8),(3,18)\}(\text{mod }(-,21))\cup R_{0,1}$
$\{(0,0),(1,3),(3,13)\}(\text{mod }(-,21))\cup R_{2,2}$	$\{(1,0),(2,6),(3,1)\}(\text{mod }(-,21))\cup R_{0,2}$
$\{(0,0),(1,17),(3,5)\}(\text{mod }(-,21))\cup R_{2,3}$	$\{(1,0),(2,3),(3,17)\}(\text{mod }(-,21))\cup R_{0,3}$
$\{(0,0),(1,9),(3,15)\}(\text{mod }(-,21))\cup R_{2,4}$	$\{(1,0),(2,10),(3,14)\}(\text{mod }(-,21))\cup R_{0,4}$
$\{(0,0),(1,14),(3,1)\}(\text{mod }(-,21))\cup R_{2,5}$	$\{(1,0),(2,13),(3,20)\}(\text{mod }(-,21))\cup R_{0,5}$
$\{(0,0),(1,12),(3,16)\}(\text{mod }(-,21))\cup R_{2,6}$	$\{(1,0),(2,5),(3,13)\}(\text{mod }(-,21))\cup R_{0,6}$
$\{(0,0),(1,8),(3,2)\}(\text{mod }(-,21))\cup R_{2,7}$	$\{(1,0),(2,15),(3,7)\}(\text{mod }(-,21))\cup R_{0,7}$
$\{(0,0),(1,18),(3,9)\}(\text{mod }(-,21))\cup R_{2,8}$	$\{(1,0),(2,9),(3,3)\}(\text{mod }(-,21))\cup R_{0,8}$

The last both parallel classes of triples are given by $\bigcup_{i=0}^{3} R_{i,9}$ and $\bigcup_{i=0}^{3} R_{i,10}$. □

Example A.9. There exists a URD($\hat{K}$; 132) with $r_4 = 5$.

Proof. Let Z_λ be the group of residues modulo λ. The design is constructed on $X = Z_4 \times Z_{33}$. Take the following five parallel classes with blocks of size four:

$$P_1 = \{(0,0),(1,0),(2,0),(3,0)\}\ (\text{mod }(-,33))$$
$$P_2 = \{(0,0),(1,1),(2,2),(3,3)\}\ (\text{mod }(-,33))$$
$$P_3 = \{(0,3),(1,2),(2,1),(3,0)\}\ (\text{mod }(-,33))$$
$$P_4 = \{(0,0),(1,2),(2,4),(3,6)\}\ (\text{mod }(-,33))$$
$$P_5 = \{(0,6),(1,4),(2,2),(3,0)\}\ (\text{mod }(-,33)).$$

It is well known that there is an RPBD(3; 33) with 16 parallel classes. Place a copy of this design on each Z_{33} set. Denote the resolution classes by $R_{i,j}$ where $i \in Z_4$ denotes on which copy of Z_{33} the parallel class is placed and $j = 1, \ldots, 16$ are the 16 resolution classes. The parallel classes of the triples are formed as follows:

$\{(0,0),(1,23),(2,32)\}(\text{mod }(-,33))\cup R_{3,1}$	$\{(0,0),(2,21),(3,16)\}(\text{mod }(-,33))\cup R_{1,1}$
$\{(0,0),(1,24),(2,3)\}(\text{mod }(-,33))\cup R_{3,2}$	$\{(0,0),(2,15),(3,4)\}(\text{mod }(-,33))\cup R_{1,2}$
$\{(0,0),(1,20),(2,7)\}(\text{mod }(-,33))\cup R_{3,3}$	$\{(0,0),(2,5),(3,28)\}(\text{mod }(-,33))\cup R_{1,3}$
$\{(0,0),(1,25),(2,10)\}(\text{mod }(-,33))\cup R_{3,4}$	$\{(0,0),(2,26),(3,13)\}(\text{mod }(-,33))\cup R_{1,4}$
$\{(0,0),(1,13),(2,19)\}(\text{mod }(-,33))\cup R_{3,5}$	$\{(0,0),(2,28),(3,31)\}(\text{mod }(-,33))\cup R_{1,5}$
$\{(0,0),(1,27),(2,23)\}(\text{mod }(-,33))\cup R_{3,6}$	$\{(0,0),(2,13),(3,26)\}(\text{mod }(-,33))\cup R_{1,6}$
$\{(0,0),(1,17),(2,20)\}(\text{mod }(-,33))\cup R_{3,7}$	$\{(0,0),(2,8),(3,25)\}(\text{mod }(-,33))\cup R_{1,7}$
$\{(0,0),(1,16),(2,9)\}(\text{mod }(-,33))\cup R_{3,8}$	$\{(0,0),(2,27),(3,8)\}(\text{mod }(-,33))\cup R_{1,8}$
$\{(0,0),(1,22),(2,14)\}(\text{mod }(-,33))\cup R_{3,9}$	$\{(0,0),(2,17),(3,21)\}(\text{mod }(-,33))\cup R_{1,9}$
$\{(0,0),(1,9),(2,25)\}(\text{mod }(-,33))\cup R_{3,10}$	$\{(0,0),(2,1),(3,17)\}(\text{mod }(-,33))\cup R_{1,10}$
$\{(0,0),(1,15),(2,6)\}(\text{mod }(-,33))\cup R_{3,11}$	$\{(0,0),(2,22),(3,7)\}(\text{mod }(-,33))\cup R_{1,11}$
$\{(0,0),(1,3),(2,30)\}(\text{mod }(-,33))\cup R_{3,12}$	$\{(0,0),(2,12),(3,18)\}(\text{mod }(-,33))\cup R_{1,12}$
$\{(0,0),(1,11),(2,18)\}(\text{mod }(-,33))\cup R_{3,13}$	$\{(0,0),(2,16),(3,12)\}(\text{mod }(-,33))\cup R_{1,13}$
$\{(0,0),(1,29),(2,24)\}(\text{mod }(-,33))\cup R_{3,14}$	$\{(0,0),(2,11),(3,19)\}(\text{mod }(-,33))\cup R_{1,14}$

$\{(0,0),(1,19),(3,14)\}(\text{mod }(-,33))\cup R_{2,1}$	$\{(1,0),(2,5),(3,10)\}(\text{mod }(-,33))\cup R_{0,1}$
$\{(0,0),(1,30),(3,10)\}(\text{mod }(-,33))\cup R_{2,2}$	$\{(1,0),(2,17),(3,11)\}(\text{mod }(-,33))\cup R_{0,2}$
$\{(0,0),(1,6),(3,22)\}(\text{mod }(-,33))\cup R_{2,3}$	$\{(1,0),(2,10),(3,7)\}(\text{mod }(-,33))\cup R_{0,3}$
$\{(0,0),(1,26),(3,2)\}(\text{mod }(-,33))\cup R_{2,4}$	$\{(1,0),(2,22),(3,1)\}(\text{mod }(-,33))\cup R_{0,4}$
$\{(0,0),(1,7),(3,1)\}(\text{mod }(-,33))\cup R_{2,5}$	$\{(1,0),(2,14),(3,25)\}(\text{mod }(-,33))\cup R_{0,5}$
$\{(0,0),(1,21),(3,5)\}(\text{mod }(-,33))\cup R_{2,6}$	$\{(1,0),(2,13),(3,5)\}(\text{mod }(-,33))\cup R_{0,6}$
$\{(0,0),(1,10),(3,32)\}(\text{mod }(-,33))\cup R_{2,7}$	$\{(1,0),(2,19),(3,26)\}(\text{mod }(-,33))\cup R_{0,7}$
$\{(0,0),(1,5),(3,20)\}(\text{mod }(-,33))\cup R_{2,8}$	$\{(1,0),(2,21),(3,3)\}(\text{mod }(-,33))\cup R_{0,8}$
$\{(0,0),(1,8),(3,29)\}(\text{mod }(-,33))\cup R_{2,9}$	$\{(1,0),(2,11),(3,32)\}(\text{mod }(-,33))\cup R_{0,9}$
$\{(0,0),(1,18),(3,9)\}(\text{mod }(-,33))\cup R_{2,10}$	$\{(1,0),(2,15),(3,8)\}(\text{mod }(-,33))\cup R_{0,10}$
$\{(0,0),(1,12),(3,24)\}(\text{mod }(-,33))\cup R_{2,11}$	$\{(1,0),(2,23),(3,14)\}(\text{mod }(-,33))\cup R_{0,11}$
$\{(0,0),(1,28),(3,15)\}(\text{mod }(-,33))\cup R_{2,12}$	$\{(1,0),(2,8),(3,18)\}(\text{mod }(-,33))\cup R_{0,12}$
$\{(0,0),(1,4),(3,23)\}(\text{mod }(-,33))\cup R_{2,13}$	$\{(1,0),(2,4),(3,23)\}(\text{mod }(-,33))\cup R_{0,13}$
$\{(0,0),(1,14),(3,11)\}(\text{mod }(-,33))\cup R_{2,14}$	$\{(1,0),(2,30),(3,6)\}(\text{mod }(-,33))\cup R_{0,14}$

The last both parallel classes of triples are given by $\bigcup_{i=0}^{3} R_{i,15}$ and $\bigcup_{i=0}^{3} R_{i,16}$. □

Example A.10. There exists a URD($\hat{K}$; 156) with $r_4 = 5$.

Proof. Let Z_λ be the group of residues modulo λ. The design is constructed on $X = Z_4 \times Z_{39}$. Take the following five parallel classes with blocks of size four:

$$P_1 = \{(0,0),(1,0),(2,0),(3,0)\} \pmod{(-,39)}$$
$$P_2 = \{(0,0),(1,1),(2,2),(3,3)\} \pmod{(-,39)}$$
$$P_3 = \{(0,3),(1,2),(2,1),(3,0)\} \pmod{(-,39)}$$
$$P_4 = \{(0,0),(1,2),(2,4),(3,6)\} \pmod{(-,39)}$$
$$P_5 = \{(0,6),(1,4),(2,2),(3,0)\} \pmod{(-,39)}.$$

It is well known that there is an RPBD(3; 39) with 19 parallel classes. Place a copy of this design on each Z_{39} set. Denote the resolution classes by $R_{i,j}$ where $i \in Z_4$ denotes on which copy of Z_{39} the parallel class is placed and $j = 1, \dots, 19$ are the resolution classes. The parallel classes of the triples are formed as follows:

$\{(0,0),(1,3),(2,19)\} \pmod{(-,39)} \cup R_{3,1}$	$\{(0,0),(1,14),(3,1)\} \pmod{(-,39)} \cup R_{2,1}$
$\{(0,0),(1,4),(2,23)\} \pmod{(-,39)} \cup R_{3,2}$	$\{(0,0),(1,12),(3,23)\} \pmod{(-,39)} \cup R_{2,2}$
$\{(0,0),(1,5),(2,22)\} \pmod{(-,39)} \cup R_{3,3}$	$\{(0,0),(1,30),(3,13)\} \pmod{(-,39)} \cup R_{2,3}$
$\{(0,0),(1,6),(2,26)\} \pmod{(-,39)} \cup R_{3,4}$	$\{(0,0),(1,23),(3,2)\} \pmod{(-,39)} \cup R_{2,4}$
$\{(0,0),(1,34),(2,13)\} \pmod{(-,39)} \cup R_{3,5}$	$\{(0,0),(1,20),(3,17)\} \pmod{(-,39)} \cup R_{2,5}$
$\{(0,0),(1,13),(2,28)\} \pmod{(-,39)} \cup R_{3,6}$	$\{(0,0),(1,26),(3,34)\} \pmod{(-,39)} \cup R_{2,6}$
$\{(0,0),(1,31),(2,38)\} \pmod{(-,39)} \cup R_{3,7}$	$\{(0,0),(1,24),(3,15)\} \pmod{(-,39)} \cup R_{2,7}$
$\{(0,0),(1,35),(2,27)\} \pmod{(-,39)} \cup R_{3,8}$	$\{(0,0),(1,27),(3,21)\} \pmod{(-,39)} \cup R_{2,8}$
$\{(0,0),(1,15),(2,20)\} \pmod{(-,39)} \cup R_{3,9}$	$\{(0,0),(1,8),(3,14)\} \pmod{(-,39)} \cup R_{2,9}$
$\{(0,0),(1,29),(2,1)\} \pmod{(-,39)} \cup R_{3,10}$	$\{(0,0),(1,10),(3,19)\} \pmod{(-,39)} \cup R_{2,10}$
$\{(0,0),(1,36),(2,11)\} \pmod{(-,39)} \cup R_{3,11}$	$\{(0,0),(1,28),(3,18)\} \pmod{(-,39)} \cup R_{2,11}$
$\{(0,0),(1,21),(2,10)\} \pmod{(-,39)} \cup R_{3,12}$	$\{(0,0),(1,16),(3,5)\} \pmod{(-,39)} \cup R_{2,12}$
$\{(0,0),(1,18),(2,14)\} \pmod{(-,39)} \cup R_{3,13}$	$\{(0,0),(1,11),(3,38)\} \pmod{(-,39)} \cup R_{2,13}$
$\{(0,0),(1,22),(2,30)\} \pmod{(-,39)} \cup R_{3,14}$	$\{(0,0),(1,33),(3,32)\} \pmod{(-,39)} \cup R_{2,14}$
$\{(0,0),(1,7),(2,33)\} \pmod{(-,39)} \cup R_{3,15}$	$\{(0,0),(1,19),(3,22)\} \pmod{(-,39)} \cup R_{2,15}$
$\{(0,0),(1,17),(2,29)\} \pmod{(-,39)} \cup R_{3,16}$	$\{(0,0),(1,25),(3,9)\} \pmod{(-,39)} \cup R_{2,16}$
$\{(0,0),(1,9),(2,34)\} \pmod{(-,39)} \cup R_{3,17}$	$\{(0,0),(1,32),(3,24)\} \pmod{(-,39)} \cup R_{2,17}$

$\{(0,0),(2,25),(3,30)\} \pmod{(-,39)} \cup R_{1,1}$	$\{(1,0),(2,9),(3,1)\} \pmod{(-,39)} \cup R_{0,1}$
$\{(0,0),(2,8),(3,35)\} \pmod{(-,39)} \cup R_{1,2}$	$\{(1,0),(2,13),(3,25)\} \pmod{(-,39)} \cup R_{0,2}$
$\{(0,0),(2,15),(3,37)\} \pmod{(-,39)} \cup R_{1,3}$	$\{(1,0),(2,27),(3,12)\} \pmod{(-,39)} \cup R_{0,3}$
$\{(0,0),(2,7),(3,25)\} \pmod{(-,39)} \cup R_{1,4}$	$\{(1,0),(2,34),(3,24)\} \pmod{(-,39)} \cup R_{0,4}$
$\{(0,0),(2,9),(3,4)\} \pmod{(-,39)} \cup R_{1,5}$	$\{(1,0),(2,33),(3,17)\} \pmod{(-,39)} \cup R_{0,5}$
$\{(0,0),(2,3),(3,10)\} \pmod{(-,39)} \cup R_{1,6}$	$\{(1,0),(2,23),(3,20)\} \pmod{(-,39)} \cup R_{0,6}$
$\{(0,0),(2,6),(3,31)\} \pmod{(-,39)} \cup R_{1,7}$	$\{(1,0),(2,3),(3,13)\} \pmod{(-,39)} \cup R_{0,7}$
$\{(0,0),(2,12),(3,29)\} \pmod{(-,39)} \cup R_{1,8}$	$\{(1,0),(2,4),(3,34)\} \pmod{(-,39)} \cup R_{0,8}$
$\{(0,0),(2,5),(3,8)\} \pmod{(-,39)} \cup R_{1,9}$	$\{(1,0),(2,22),(3,15)\} \pmod{(-,39)} \cup R_{0,9}$
$\{(0,0),(2,16),(3,12)\} \pmod{(-,39)} \cup R_{1,10}$	$\{(1,0),(2,30),(3,7)\} \pmod{(-,39)} \cup R_{0,10}$
$\{(0,0),(2,17),(3,11)\} \pmod{(-,39)} \cup R_{1,11}$	$\{(1,0),(2,6),(3,21)\} \pmod{(-,39)} \cup R_{0,11}$
$\{(0,0),(2,24),(3,28)\} \pmod{(-,39)} \cup R_{1,12}$	$\{(1,0),(2,24),(3,5)\} \pmod{(-,39)} \cup R_{0,12}$
$\{(0,0),(2,31),(3,20)\} \pmod{(-,39)} \cup R_{1,13}$	$\{(1,0),(2,32),(3,14)\} \pmod{(-,39)} \cup R_{0,13}$
$\{(0,0),(2,21),(3,27)\} \pmod{(-,39)} \cup R_{1,14}$	$\{(1,0),(2,29),(3,16)\} \pmod{(-,39)} \cup R_{0,14}$
$\{(0,0),(2,32),(3,7)\} \pmod{(-,39)} \cup R_{1,15}$	$\{(1,0),(2,10),(3,19)\} \pmod{(-,39)} \cup R_{0,15}$
$\{(0,0),(2,18),(3,26)\} \pmod{(-,39)} \cup R_{1,16}$	$\{(1,0),(2,36),(3,10)\} \pmod{(-,39)} \cup R_{0,16}$
$\{(0,0),(2,36),(3,16)\} \pmod{(-,39)} \cup R_{1,17}$	$\{(1,0),(2,21),(3,32)\} \pmod{(-,39)} \cup R_{0,17}$

The last both parallel classes of triples are given by $\bigcup_{i=0}^{3} R_{i,18}$ and $\bigcup_{i=0}^{3} R_{i,19}$. □

Example A.11. There exists a URD($\hat{K}$; 204) with $r_4 = 5$.

Proof. Let Z_λ be the group of residues modulo λ. The design is constructed on $X = Z_4 \times Z_{51}$. Take the following five parallel classes with blocks of size four:

$$P_1 = \{(0,3),(1,2),(2,1),(3,0)\} \pmod{(-,51)}$$
$$P_2 = \{(0,0),(1,2),(2,4),(3,6)\} \pmod{(-,51)}$$
$$P_3 = \{(0,6),(1,4),(2,2),(3,0)\} \pmod{(-,51)}$$
$$P_4 = \{(0,0),(1,3),(2,6),(3,9)\} \pmod{(-,51)}$$
$$P_5 = \{(0,9),(1,6),(2,3),(3,0)\} \pmod{(-,51)}.$$

It is well known that there is an RPBD(3; 51) with 25 parallel classes. Place a copy of this design on each Z_{51} set. Denote the resolution classes by $R_{i,j}$ where $i \in Z_4$ denotes on which copy of Z_{51} the parallel class is placed and $j = 1, \dots, 25$ are the resolution classes. The parallel classes of the triples are formed as follows:

$\{(0,0),(1,38),(2,26)\}(\text{mod }(-,51)) \cup R_{3,1}$	$\{(0,0),(1,41),(3,36)\}(\text{mod }(-,51)) \cup R_{2,1}$
$\{(0,0),(1,15),(2,37)\}(\text{mod }(-,51)) \cup R_{3,2}$	$\{(0,0),(1,47),(3,32)\}(\text{mod }(-,51)) \cup R_{2,2}$
$\{(0,0),(1,36),(2,42)\}(\text{mod }(-,51)) \cup R_{3,3}$	$\{(0,0),(1,22),(3,3)\}(\text{mod }(-,51)) \cup R_{2,3}$
$\{(0,0),(1,8),(2,50)\}(\text{mod }(-,51)) \cup R_{3,4}$	$\{(0,0),(1,39),(3,38)\}(\text{mod }(-,51)) \cup R_{2,4}$
$\{(0,0),(1,0),(2,13)\}(\text{mod }(-,51)) \cup R_{3,5}$	$\{(0,0),(1,45),(3,22)\}(\text{mod }(-,51)) \cup R_{2,5}$
$\{(0,0),(1,21),(2,36)\}(\text{mod }(-,51)) \cup R_{3,6}$	$\{(0,0),(1,18),(3,26)\}(\text{mod }(-,51)) \cup R_{2,6}$
$\{(0,0),(1,14),(2,7)\}(\text{mod }(-,51)) \cup R_{3,7}$	$\{(0,0),(1,29),(3,17)\}(\text{mod }(-,51)) \cup R_{2,7}$
$\{(0,0),(1,5),(2,1)\}(\text{mod }(-,51)) \cup R_{3,8}$	$\{(0,0),(1,34),(3,16)\}(\text{mod }(-,51)) \cup R_{2,8}$
$\{(0,0),(1,17),(2,18)\}(\text{mod }(-,51)) \cup R_{3,9}$	$\{(0,0),(1,31),(3,21)\}(\text{mod }(-,51)) \cup R_{2,9}$
$\{(0,0),(1,42),(2,0)\}(\text{mod }(-,51)) \cup R_{3,10}$	$\{(0,0),(1,28),(3,4)\}(\text{mod }(-,51)) \cup R_{2,10}$
$\{(0,0),(1,33),(2,40)\}(\text{mod }(-,51)) \cup R_{3,11}$	$\{(0,0),(1,30),(3,50)\}(\text{mod }(-,51)) \cup R_{2,11}$
$\{(0,0),(1,23),(2,43)\}(\text{mod }(-,51)) \cup R_{3,12}$	$\{(0,0),(1,37),(3,46)\}(\text{mod }(-,51)) \cup R_{2,12}$
$\{(0,0),(1,46),(2,25)\}(\text{mod }(-,51)) \cup R_{3,13}$	$\{(0,0),(1,7),(3,20)\}(\text{mod }(-,51)) \cup R_{2,13}$
$\{(0,0),(1,10),(2,10)\}(\text{mod }(-,51)) \cup R_{3,14}$	$\{(0,0),(1,16),(3,8)\}(\text{mod }(-,51)) \cup R_{2,14}$
$\{(0,0),(1,1),(2,29)\}(\text{mod }(-,51)) \cup R_{3,15}$	$\{(0,0),(1,11),(3,41)\}(\text{mod }(-,51)) \cup R_{2,15}$
$\{(0,0),(1,13),(2,32)\}(\text{mod }(-,51)) \cup R_{3,16}$	$\{(0,0),(1,24),(3,25)\}(\text{mod }(-,51)) \cup R_{2,16}$
$\{(0,0),(1,19),(2,3)\}(\text{mod }(-,51)) \cup R_{3,17}$	$\{(0,0),(1,6),(3,29)\}(\text{mod }(-,51)) \cup R_{2,17}$
$\{(0,0),(1,43),(2,23)\}(\text{mod }(-,51)) \cup R_{3,18}$	$\{(0,0),(1,32),(3,37)\}(\text{mod }(-,51)) \cup R_{2,18}$
$\{(0,0),(1,12),(2,24)\}(\text{mod }(-,51)) \cup R_{3,19}$	$\{(0,0),(1,27),(3,44)\}(\text{mod }(-,51)) \cup R_{2,19}$
$\{(0,0),(1,25),(2,46)\}(\text{mod }(-,51)) \cup R_{3,20}$	$\{(0,0),(1,35),(3,24)\}(\text{mod }(-,51)) \cup R_{2,20}$
$\{(0,0),(1,9),(2,35)\}(\text{mod }(-,51)) \cup R_{3,21}$	$\{(0,0),(1,40),(3,5)\}(\text{mod }(-,51)) \cup R_{2,21}$
$\{(0,0),(1,26),(2,16)\}(\text{mod }(-,51)) \cup R_{3,22}$	$\{(0,0),(1,44),(3,18)\}(\text{mod }(-,51)) \cup R_{2,22}$
$\{(0,0),(1,4),(2,41)\}(\text{mod }(-,51)) \cup R_{3,23}$	$\{(0,0),(1,20),(3,35)\}(\text{mod }(-,51)) \cup R_{2,23}$

$\{(0,0),(2,14),(3,30)\}(\text{mod }(-,51)) \cup R_{1,1}$	$\{(1,0),(2,18),(3,19)\}(\text{mod }(-,51)) \cup R_{0,1}$
$\{(0,0),(2,33),(3,7)\}(\text{mod }(-,51)) \cup R_{1,2}$	$\{(1,0),(2,14),(3,21)\}(\text{mod }(-,51)) \cup R_{0,2}$
$\{(0,0),(2,15),(3,15)\}(\text{mod }(-,51)) \cup R_{1,3}$	$\{(1,0),(2,16),(3,2)\}(\text{mod }(-,51)) \cup R_{0,3}$
$\{(0,0),(2,28),(3,47)\}(\text{mod }(-,51)) \cup R_{1,4}$	$\{(1,0),(2,45),(3,7)\}(\text{mod }(-,51)) \cup R_{0,4}$
$\{(0,0),(2,34),(3,19)\}(\text{mod }(-,51)) \cup R_{1,5}$	$\{(1,0),(2,34),(3,24)\}(\text{mod }(-,51)) \cup R_{0,5}$
$\{(0,0),(2,38),(3,34)\}(\text{mod }(-,51)) \cup R_{1,6}$	$\{(1,0),(2,11),(3,31)\}(\text{mod }(-,51)) \cup R_{0,6}$
$\{(0,0),(2,9),(3,0)\}(\text{mod }(-,51)) \cup R_{1,7}$	$\{(1,0),(2,43),(3,38)\}(\text{mod }(-,51)) \cup R_{0,7}$
$\{(0,0),(2,5),(3,13)\}(\text{mod }(-,51)) \cup R_{1,8}$	$\{(1,0),(2,5),(3,14)\}(\text{mod }(-,51)) \cup R_{0,8}$
$\{(0,0),(2,30),(3,1)\}(\text{mod }(-,51)) \cup R_{1,9}$	$\{(1,0),(2,10),(3,34)\}(\text{mod }(-,51)) \cup R_{0,9}$
$\{(0,0),(2,39),(3,33)\}(\text{mod }(-,51)) \cup R_{1,10}$	$\{(1,0),(2,25),(3,18)\}(\text{mod }(-,51)) \cup R_{0,10}$
$\{(0,0),(2,27),(3,11)\}(\text{mod }(-,51)) \cup R_{1,11}$	$\{(1,0),(2,38),(3,26)\}(\text{mod }(-,51)) \cup R_{0,11}$
$\{(0,0),(2,12),(3,39)\}(\text{mod }(-,51)) \cup R_{1,12}$	$\{(1,0),(2,46),(3,12)\}(\text{mod }(-,51)) \cup R_{0,12}$
$\{(0,0),(2,19),(3,2)\}(\text{mod }(-,51)) \cup R_{1,13}$	$\{(1,0),(2,8),(3,22)\}(\text{mod }(-,51)) \cup R_{0,13}$
$\{(0,0),(2,8),(3,31)\}(\text{mod }(-,51)) \cup R_{1,14}$	$\{(1,0),(2,17),(3,48)\}(\text{mod }(-,51)) \cup R_{0,14}$
$\{(0,0),(2,31),(3,10)\}(\text{mod }(-,51)) \cup R_{1,15}$	$\{(1,0),(2,29),(3,11)\}(\text{mod }(-,51)) \cup R_{0,15}$
$\{(0,0),(2,20),(3,12)\}(\text{mod }(-,51)) \cup R_{1,16}$	$\{(1,0),(2,4),(3,44)\}(\text{mod }(-,51)) \cup R_{0,16}$
$\{(0,0),(2,48),(3,23)\}(\text{mod }(-,51)) \cup R_{1,17}$	$\{(1,0),(2,36),(3,3)\}(\text{mod }(-,51)) \cup R_{0,17}$
$\{(0,0),(2,44),(3,14)\}(\text{mod }(-,51)) \cup R_{1,18}$	$\{(1,0),(2,32),(3,10)\}(\text{mod }(-,51)) \cup R_{0,18}$
$\{(0,0),(2,11),(3,43)\}(\text{mod }(-,51)) \cup R_{1,19}$	$\{(1,0),(2,23),(3,35)\}(\text{mod }(-,51)) \cup R_{0,19}$
$\{(0,0),(2,22),(3,28)\}(\text{mod }(-,51)) \cup R_{1,20}$	$\{(1,0),(2,40),(3,0)\}(\text{mod }(-,51)) \cup R_{0,20}$
$\{(0,0),(2,17),(3,27)\}(\text{mod }(-,51)) \cup R_{1,21}$	$\{(1,0),(2,27),(3,42)\}(\text{mod }(-,51)) \cup R_{0,21}$
$\{(0,0),(2,21),(3,49)\}(\text{mod }(-,51)) \cup R_{1,22}$	$\{(1,0),(2,24),(3,29)\}(\text{mod }(-,51)) \cup R_{0,22}$
$\{(0,0),(2,2),(3,40)\}(\text{mod }(-,51)) \cup R_{1,23}$	$\{(1,0),(2,33),(3,37)\}(\text{mod }(-,51)) \cup R_{0,23}$

The last both parallel classes of triples are given by $\bigcup_{i=0}^{3} R_{i,24}$ and $\bigcup_{i=0}^{3} R_{i,25}$. □

Example A.12. An $\mathrm{LUGDD}_4(3, 4; 12)$, $\boldsymbol{G} = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}\}$; each row forms a parallel class:

(2 7 11; 2 1 3), (1 5 9; 0 3 3), (3 8 12; 2 3 1), (4 6 10; 2 2 0),
(3 8 11; 1 1 0), (4 5 12; 0 1 1), (2 7 9; 0 0 0), (1 6 10; 0 3 3),
(2 8 11; 0 3 3), (3 5 10; 1 3 2), (1 7 12; 3 3 0), (4 6 9; 0 3 3),
(1 5 11; 3 3 0), (4 8 10; 3 1 2), (3 7 9; 2 0 2), (2 6 12; 0 1 1),
(1 8 9; 3 0 1), (3 6 12; 3 1 2), (4 7 11; 2 0 2), (2 5 10; 0 0 0),
(2 8 10; 2 1 3), (4 7 12; 3 0 1), (1 6 11; 1 1 0), (3 5 9; 0 2 2),

(4 5 11; 1 3 2), (2 6 9; 2 3 1), (3 7 10; 1 0 3), (1 8 12; 0 2 2),
(4 8 9; 1 0 3), (2 5 12; 1 0 3), (1 7 10; 0 2 2), (3 6 11; 2 3 1),
(3 6 11; 1 0 3), (1 5 9; 1 1 0), (4 7 12; 1 3 2), (2 8 10; 1 2 1),
(3 7 12; 3 2 3), (4 8 9; 2 2 0), (1 6 11; 2 0 2), (2 5 10; 2 3 1),
(3 6 12; 0 0 0), (2 7 9; 3 2 3), (4 5 11; 3 2 3), (1 8 10; 1 1 0),
(3 5 9; 2 3 1), (2 8 12; 3 3 0), (1 7 11; 2 2 0), (4 6 10; 1 3 2),
(1 5 12; 2 0 2), (4 7 10; 0 0 0), (3 8 11; 0 2 2), (2 6 9; 1 1 0),
(2 6 12; 3 2 3), (4 8 11; 0 1 1), (3 5 10; 3 2 3), (1 7 9; 1 2 1),
(3 8 9; 3 1 2), (1 6 10; 3 0 1), (2 7 11; 1 2 1), (4 5 12; 2 2 0),
(1 8 12; 2 1 3), (4 6 9; 3 1 2), (2 5 11; 3 0 1), (3 7 10; 0 1 1).

Example A.13. An $\mathrm{LUGDD}_4(3, 3; 12)$, $\boldsymbol{G} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}\}$; each row forms a parallel class:

(2 4 11; 2 1 3), (1 8 12; 3 0 1), (3 5 9; 2 2 0), (6 7 10; 1 2 1),
(1 5 9; 0 1 1), (2 4 10; 0 1 1), (3 8 12; 2 2 0), (6 7 11; 2 0 2),
(2 9 11; 3 0 1), (3 4 7; 3 2 3), (5 8 10; 2 1 3), (1 6 12; 3 3 0),
(2 7 12; 2 3 1), (1 4 8; 2 0 2), (6 9 10; 3 3 0), (3 5 11; 3 3 0),
(2 5 12; 3 1 2), (1 7 10; 1 0 3), (3 6 9; 2 3 1), (4 8 11; 3 1 2),
(2 6 8; 3 3 0), (3 4 10; 2 0 2), (5 7 12; 0 3 3), (1 9 11; 0 2 2),
(1 5 10; 2 2 0), (3 7 11; 1 2 1), (4 9 12; 1 2 1), (2 6 8; 2 0 2),
(3 8 10; 3 1 2), (2 5 7; 0 1 1), (1 6 11; 2 1 3), (4 9 12; 2 0 2),
(3 6 12; 0 3 3), (2 9 10; 2 3 1), (1 4 7; 0 0 0), (5 8 11; 1 2 1),
(2 5 7; 1 3 2), (3 4 11; 1 1 0), (6 8 12; 3 1 2), (1 9 10; 3 1 2),
(4 8 12; 0 3 3), (1 5 7; 3 2 3), (3 6 11; 3 0 1), (2 9 10; 1 0 3),
(2 5 12; 2 2 0), (6 7 10; 0 0 0), (1 8 11; 1 0 3), (3 4 9; 0 0 0),
(3 5 10; 1 3 2), (4 7 11; 2 2 0), (1 9 12; 2 1 3), (2 6 8; 0 1 1),
(3 5 8; 0 0 0), (1 4 10; 3 3 0), (6 9 12; 2 2 0), (2 7 11; 0 3 3),
(5 9 11; 3 3 0), (1 6 7; 0 3 3), (3 8 10; 1 2 1), (2 4 12; 3 0 1),
(1 4 8; 1 2 1), (2 6 10; 1 2 1), (3 7 12; 0 0 0), (5 9 11; 2 1 3),
(1 6 11; 1 3 2), (5 8 10; 3 3 0), (2 4 9; 1 0 3), (3 7 12; 3 1 2),
(3 6 9; 1 1 0), (1 5 12; 1 2 1), (4 7 10; 1 3 2), (2 8 11; 2 2 0).

Example A.14. An $\mathrm{LUGDD}_4(\hat{K}, 3; 12)$ with $r_4 = 2$, $\boldsymbol{G} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}\}$; each row forms a parallel class:

(2 4 7; 0 2 2), (3 5 12; 1 2 1), (6 9 11; 0 1 1), (1 8 10; 3 0 1),
(2 5 12; 0 0 0), (1 8 11; 1 0 3), (3 6 9; 1 0 3), (4 7 10; 3 2 3),
(2 4 11; 1 1 0), (3 5 9; 3 3 0), (6 8 10; 1 0 3), (1 7 12; 2 2 0),
(2 5 10; 2 2 0), (3 9 11; 1 1 0), (4 8 12; 0 3 3), (1 6 7; 0 3 3),
(2 9 12; 3 1 2), (1 4 10; 2 3 1), (3 6 8; 3 3 0), (5 7 11; 3 2 3),
(1 5 10; 0 1 1), (2 6 8; 2 1 3), (3 7 12; 1 3 2), (4 9 11; 0 3 3),
(1 4 9; 3 2 3), (2 5 11; 1 2 1), (3 7 10; 0 2 2), (6 8 12; 2 2 0),
(2 8 11; 3 0 1), (3 4 7; 3 3 0), (5 9 12; 1 2 1), (1 6 10; 3 2 3),
(2 6 11; 3 3 0), (1 5 7; 2 0 2), (3 9 10; 2 3 1), (4 8 12; 2 0 2),
(3 4 11; 1 3 2), (2 5 8; 3 0 1), (6 7 10; 1 2 1), (1 9 12; 0 3 3),
(2 7 10; 0 0 0), (1 5 8; 1 0 3), (3 6 11; 2 0 2), (4 9 12; 2 2 0),
(3 5 8; 0 2 2), (1 4 12; 0 1 1), (6 7 11; 2 3 1), (2 9 10; 1 3 2),
(2 8 12; 2 3 1), (1 6 9; 2 3 1), (3 4 10; 0 0 0), (5 7 11; 1 3 2),
(3 6 12; 0 0 0), (2 4 9; 3 0 1), (1 7 11; 1 1 0), (5 8 10; 0 2 2),
(2 4 7; 2 3 1), (1 6 12; 1 0 3), (3 8 11; 0 2 2), (5 9 10; 3 3 0),
(1 5 9 11; 3 1 3 2 0 2), (2 6 7 12; 1 1 2 0 1 1), (3 4 8 10; 2 1 1 3 3 0),
(1 4 8 11; 1 2 2 1 1 0), (2 6 9 10; 0 2 1 2 1 3), (3 5 7 12; 2 2 1 0 3 3).

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